a) Estimation

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - C_k\hat{x}_{k|k-1})$$

$$K_k = \Sigma_{k|k-1} C_k^T [C_k \Sigma_{k|k-1} C_k^T + \bar{R}_k]^{-1} = \Sigma_{k|k} C_k^T \bar{R}_k^{-1}$$

$$\Sigma_{k|k} = (I - K_k C_k)\Sigma_{k|k-1}$$

Dans notre cas

$$\phi_k = 0.5$$
$$E_k = 2$$
$$C_k = 1$$
$$\bar{Q}_k = 1$$
$$\bar{R}_k = 3$$

donc $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - \hat{x}_{k|k-1})$

$$K_k = \Sigma_{k|k-1}(\Sigma_{k|k-1} + 3)^{-1} = \frac{1}{3}\Sigma_{k|k}$$

$$\Sigma_{k|k} = (1 - K_k)\Sigma_{k|k-1}$$

Prédiction

$$\hat{x}_{k+1|k} = \phi_k \hat{x}_{k|k}$$

$$\Sigma_{k+1|k} = \phi_k \Sigma_{k|k} \phi_k^T + E_k \bar{Q}_k E_k^T$$

donc $\hat{x}_{k+1|k} = 0.5\,\hat{x}_{k|k}$

$$\Sigma_{k+1|k} = \frac{1}{4}\Sigma_{k|k} + 4$$

Forme condensée ②

$$\hat{x}_k = \cancel{\Phi}_{k-1}\, \hat{x}_{k-1} + K_k\left(y_k - C_k \cancel{\Phi}_{k-1}\, \hat{x}_{k-1}\right)$$

$$= 0.5\,\hat{x}_{k-1} + \frac{\Sigma_{k|k-1}}{\Sigma_{k|k-1}+3}\left(y_k - 0{,}5\,\hat{x}_{k-1}\right)$$

avec $\Sigma_{k|k-1} = \frac{1}{4}\,\Sigma_{k-1|k-1} + 4$

$$= \frac{1}{4}\left(1 - K_{k-1}\right)\Sigma_{k-1|k-2} + 4$$

b) A.N.

$k=0$ $\quad \hat{x}_{0|0} = \hat{x}_{0|-1} + K_0\left(y_0 - \hat{x}_{0|-1}\right)$

je pose $\hat{x}_{0|-1} = \bar{x}_0$

$$\hat{x}_{0|0} = \bar{x}_0 + K_0\left(y_0 - \bar{x}_0\right)$$

$K_0 = \Sigma_{0|-1}\left(\Sigma_{0|-1} + 3\right)^{-1}$ $\quad$ or $\Sigma_{0|-1} = \mathrm{Var}\{x_0\} = 3$

donc $K_0 = \frac{3}{3+3} = \frac{1}{2}$

$$\hat{x}_{0|0} = \bar{x}_0 + \frac{1}{2}\left(y_0 - \bar{x}_0\right)$$

$$= \frac{1}{2}\left(\bar{x}_0 + y_0\right)$$

$$\Sigma_{0|0} = \left(1 - K_0\right)\Sigma_{0|-1}$$

$$= \frac{1}{2} \times 3 = \frac{3}{2}$$

$\underline{k=1}$

$$\hat{x}_{1|1} = 0.5\,\hat{x}_{0|0} + K_1\left(y_1 - 0.5\,\hat{x}_{0|0}\right)$$

$$K_1 = \Sigma_{1|0}\left(\Sigma_{1|0} + 3\right)^{-1}$$

$$\Sigma_{1|0} = \frac{1}{4}\,\Sigma_{0|0} + 4$$

$$= \frac{1}{4} \times \frac{3}{2} + 4$$

$$= \frac{35}{8} = 4{,}375$$

$$K_1 = \frac{35}{8}\left(\frac{35}{8} + 3\right)^{-1}$$

$$= \frac{35}{8}\left(\frac{59}{8}\right)^{-1} = \frac{35}{59} = 0{,}59$$

$$\hat{x}_{1|1} = \frac{1}{2} \times \frac{1}{2}\left(\bar{x}_0 + y_0\right) + \frac{35}{59}\left(y_1 - \frac{1}{2} \times \frac{1}{2}\left(\bar{x}_0 + y_0\right)\right)$$

$$= \frac{1}{4}\left(\bar{x}_0 + y_0\right) + \frac{35}{59}\,y_1 - \frac{35}{59} \times \frac{1}{4}\left(\bar{x}_0 + y_0\right)$$

$$= \frac{1}{4}\left(\bar{x}_0 + y_0\right)\left(1 - \frac{35}{59}\right) + \frac{35}{59}\,y_1$$

$$= \frac{6}{59}\left(\bar{x}_0 + y_0\right) + \frac{35}{59}\,y_1 = 0{,}10\left(\bar{x}_0 + y_0\right) + 0{,}59\,y_1$$

$\underline{k=2}$

$$\Sigma_{1|1} = \left(1 - K_1\right)\Sigma_{1|0}$$

$$= \left(1 - \frac{35}{59}\right)\frac{35}{8}$$

$$= \frac{105}{59} = 1{,}78$$

$k=2$

$$\hat{x}_{2|2} = \frac{1}{2}\hat{x}_{1|1} + K_2\left(y_2 - \frac{1}{2}\hat{x}_{1|1}\right)$$

$$K_2 = \Sigma_{2|1}\left(\Sigma_{2|1} + 3\right)^{-1}$$

$$\Sigma_{2|1} = \frac{1}{4}\Sigma_{1|1} + 4$$

$$= \frac{1}{4}\,\frac{105}{59} + 4 = \frac{1049}{236} = 4{,}45$$

$$K_2 = \frac{1049}{236}\left(\frac{1049}{236} + 3\right)^{-1}$$

$$= \frac{1049}{236}\,\frac{236}{1757} = \frac{1049}{1757} = 0{,}597$$

$$\hat{x}_{2|2} = \frac{1}{2}\left[\frac{6}{59}(\bar{x}_0 + y_0) + \frac{35}{59}y_1\right] + \frac{1049}{1757}\left[y_2 - \frac{1}{2}[\quad]\right]$$

$$= \frac{1}{2}[\qquad]\left(1 - \frac{1049}{1757}\right) + \frac{1049}{1757}y_2$$

$$= \frac{1}{2}[\qquad]\frac{708}{1757} + \frac{1049}{1757}y_2$$

$$= \frac{354}{1757}\left[\frac{6}{59}(\bar{x}_0 + y_0) + \frac{35}{59}y_1\right] + \frac{1049}{1757}y_2$$

$$= \frac{36}{1757}(\bar{x}_0 + y_0) + \frac{210}{1757}y_1 + \frac{1049}{1757}y_2$$

$$= 0{,}02(\bar{x}_0 + y_0) + 0{,}12\,y_1 + 0{,}597\,y_2$$

$$\Sigma_{2|2} = (1 - K_2)\,\Sigma_{2|1}$$

$$= \left(1 - \frac{1049}{1757}\right) \times \frac{1049}{236}$$

$$= \frac{708}{1757} \times \frac{1049}{236} = \frac{3 \times 1049}{1757}$$

$$= \frac{3147}{1757} = 1{,}79$$

$k=3$

$$\hat{x}_{3|3} = \frac{1}{2}\hat{x}_{2|2} + K_3\left(y_3 - \frac{1}{2}\hat{x}_{2|2}\right)$$

$$K_3 = \Sigma_{3|2}\left(\Sigma_{3|2} + 3\right)^{-1}$$

$$\Sigma_{3|2} = \frac{1}{4}\Sigma_{2|2} + 4$$

$$= \frac{1}{4} \times \frac{3147}{1757} + 4$$

$$= \frac{31259}{7028}$$

$$K_3 = \frac{31259}{7028}\left(\frac{31259}{7028} + 3\right)^{-1}$$

$$= \frac{31259}{7028} \cdot \frac{7028}{52343} = \frac{31259}{52343} = 0{,}597$$

$$\hat{x}_{3|3} = \frac{1}{2}\left[\qquad\right] + \frac{31259}{52343}\left(y_3 - \frac{1}{2}[\quad]\right)$$

$$= \frac{18}{1757}(\bar{x}_0 + y_0) + \frac{105}{1757}y_1 + \frac{1}{2}\times\frac{1049}{1757}y_2 + \frac{31259}{52343}y_3$$

$$- \frac{31259}{52343} \times \frac{18}{1755}(\bar{x}_0 + y_0) - \frac{31259}{52343} \times \frac{105}{1757}y_1$$

$$- \frac{31259}{52343} \times \frac{1}{2}\times\frac{1049}{1757}y_2$$

$$= 0{,}004(\bar{x}_0 + y_0) + 0{,}024y_1 + 0{,}121y_2 + 0{,}597y_3$$

$$\Sigma_{3|3} = (1 - K_3)\Sigma_{3|2}$$

$$= \frac{21084}{52343} \times \frac{31259}{7028}$$

$$= 3 \times \frac{31259}{52343}$$

c) $\hat{x}_k = 0{,}5\,\hat{x}_{k-1} + 0{,}597(y_k - 0{,}5\hat{x}_{k-1})$

$= 0{,}202\,\hat{x}_{k-1} + 0{,}597\,y_k$

d) Comportement asymptotique

$\Sigma_{k+1|k+1} = \Sigma_{k|k}$ Σ devient constant et K aussi.

$\Sigma = (1-K)\Lambda$ en posant $\Lambda_k = \Sigma_{k|k-1}$

$\Sigma = \left(1 - \frac{\Lambda}{\Lambda+3}\right)\Lambda = \frac{3\Lambda}{\Lambda+3}$

or $\Lambda = \frac{1}{4}\Sigma + 4 = \frac{1}{4}\,\frac{3\Lambda}{\Lambda+3} + 4$

$4\Lambda(\Lambda+3) = 3\Lambda + 4\times 4(\Lambda+3)$

$4\Lambda^2 + 12\Lambda = 3\Lambda + 16\Lambda + 48$

$4\Lambda^2 - 7\Lambda - 48 = 0$

($\Lambda < 0$ ne convient pas)

$\Lambda = \frac{7 + \sqrt{49^2 + 16\times 48}}{8}$

$= \frac{7 + \sqrt{49 + 768}}{8}$

$= \frac{7 + \sqrt{817}}{8} = \frac{7 + 28{,}58}{8} \simeq 4{,}448$

$K = \frac{\Lambda}{\Lambda+3} \simeq 0{,}597$