# ELECTROKINETICS 2

***Summary:***

*Chapter 1: FILTERS (by myself, Marc Lethiecq) 6 x 1h20min*

*Chapiter 2: QUADRUPOLES (by Maxime Bavencoffe)*

## References


- Electronique - Théorie du signal et composants - Cours et exercices corrigés - éditions DUNOD - Manneville / Esquieu
- Les fondements du génie électrique - éditions TEC&DOC - Laurent Henry
- Electronic Devices and Circuits, Schaum's outline series – Jimmie J. Cathey

# Course organisation

*Lectures (Cours Magistral, CM) mixed with exercises (Travaux Dirigés,TD), including tests*

*+8 h Lab. Work (Travaux Pratiques, TP)*

Evaluation :

- *Tests (40%)*
- *Lab Work (20%)*
- *Final exam(40%)*

# *Electrokinetics 2 (Electrocinétique 2)*

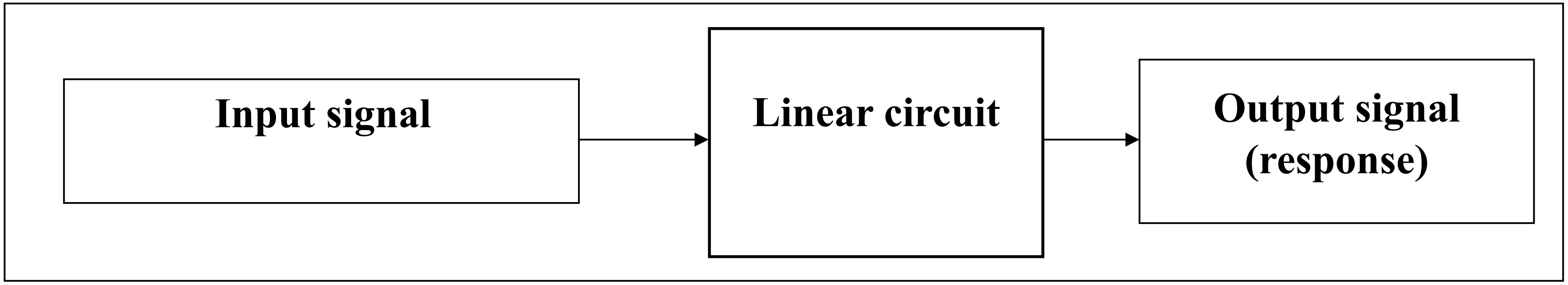
***Summary of chapter 1***

# 1.1. Frequency (harmonic) analysis of linear circuits

Here we will describe the methods used to analyse the behaviour in the frequency domain of a linear circuit, i.e. how the output varies when the input frequency changes.

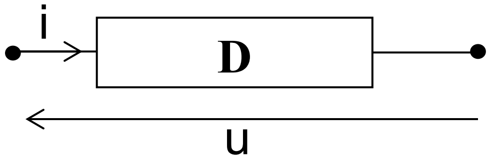
In order to study a linear circuit, we will apply a sinusoidal input signal (for instance using a function generator in sinusoidal mode) and study the output (for instance using an oscilloscope). The input, usually represented on the left side is also called the excitation signal, while the output, generally represented on the right side is also called the response.

Thanks to the linearity of the circuit, all signals will be sinusoidal and their frequency will be equal to that of the input, so we will only use **complex notations** to represent the signals (voltages and currents).

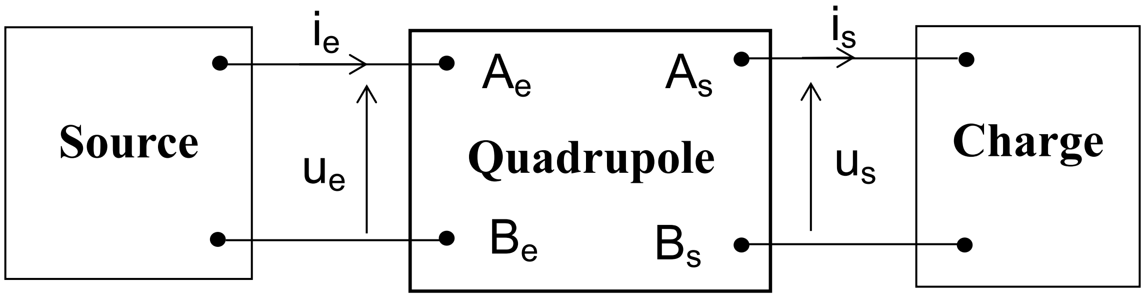


### 1.1.1. Dipoles and quadrupoles

A **dipole** is an electrical element (or component) with **two connections**:




A **quadrupole** is an electrical circuit that has **four connections**: two input connections and two output connections. There are four electrical quantities: input voltage $u_e$ and input current $i_e$, output voltage $u_s$ and output current $i_s$.




A quadrupole is made of dipoles (it can be represented as an assembly of interconnected dipoles). It is said to be linear if and only if all of the dipoles it is made of are linear.

### 1.1.2. Harmonic transfer function of a linear quadrupole

The harmonic transfer function of a linear quadrupole, generally noted ***T**(jω)* (or ***H**(jω)*) is the ratio of the complex representation of the output signal S to that of the input signal E:

$$\underline{T}(j\omega) = \underline{S} / \underline{E}$$

This function depends on the characteristics of the quadrupole and on the frequency (noted f or sometimes ν) of the signals (generally voltages, but can also be currents) which is imposed by the input. Often, instead of the frequency (in Hz), the angular frequency ω in rd/s is used, where $\omega = 2\pi f$ .

### 1.1.3. Magnitude, amplification/attenuation and phase shift

To completely characterise the response of a linear circuit to a sinusoidal input, one needs to study its transfer function T as a function of frequency f or angular frequency ω: T being a complex function, we will need to study both

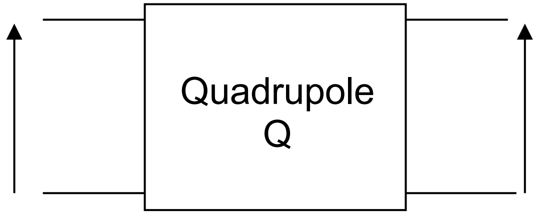
- ***its magnitude***: $A(\omega) = |\underline{T}(j\omega)|$
- ***its phase shift***: $\phi(\omega) = \arg(\underline{T}())$

If A() > 1 the circuit amplifies the input (the output amplitude is higher than that of the input)

If A() < the circuit attenuates the input (the output amplitude is lower than that of the input)

## What can be the use of a linear quadrupole?

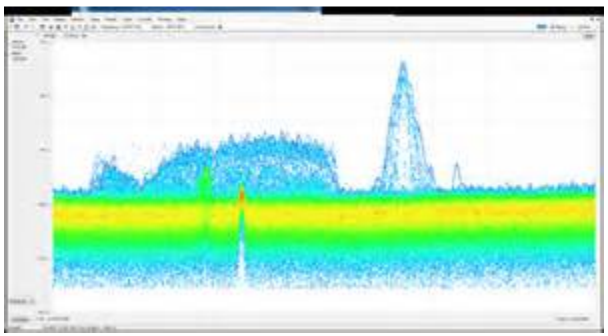
**Amplify a signal**




**Shift a signal's phase**

**Filter (sort out) a mix of several signals: spectral analysis, selection, elimination of noise…**

**Change the shape of a complex signal by acting differently on its harmonics …**



### 1.1.4. The GAIN of a quadrupole

One can define the **gain** of a quadrupole (without dimension) using the **decibel (dB)** as unit, by:

$$\mathbf{G_{dB} = 20 \log_{10}(A(\omega)) = 20 \log_{10} (|T(j\omega)|)}$$

The gain is just another way to quantify the magnitude of the transfer function.

*A few comments:*

The decibel is a sub-multiple of the bel, a unit named after Alexander Graham Bell, the inventor of the telephone. It was first - and still is - used to measure sound intensity (it takes into account the fact that the human ear has a close to logarithmic response) and has been extended to electrical quantities.

Originally the gain was was defined as a ratio of powers : $G_B = \log (P_{out}/P_{in})$ in bels, but a unit ten times smaller is now preferred: the decibel defined as $\mathbf{G_{dB} = 10 \log (P_{out}/P_{in})}$ .

Since the electrical power is proportional to the voltage (or current) square, the gain in decibels can be calculated as $G_{dB} = 10 \log (\alpha U_{out}^2/\alpha U_{in}^2) = \mathbf{20 \log (U_{out}/U_{in})}$.

**Interpretation of the gain**

1) What does it mean if the gain $G_{dB}$ positive ? Negative ?

**The gain $G_{dB}$ is positive if & only if $A > 1$ $\Rightarrow$ the output is amplified as compared to the input .**

**The gain $G_{dB}$ is negative if & only if $A < 1$ $\Rightarrow$ the output is attenuated as compared to the input .**

2) What is the gain and the transfer function's magnitude (also called the amplification) of a quadrupole if we obtain a 1V output when applying a 5V input ?

**$A = Us / Ue = 1 / 5 = 0{,}2$ and $G_{dB} = 20 \log A = 20 \log(0{,}2) = -14dB$**

3) What do the following sentences mean: « at 3 kHz, this quadrupole has a gain of +20 dB / -20 dB / -40 dB / -3 / +6 dB » ?

**$G_{dB} = +20dB \Rightarrow 20 \log (A) = +20 \Rightarrow \log (A) = 1 \Rightarrow A = 10^1 = 10$**

**$G_{dB} = -20dB \Rightarrow 20 \log (A) = -20 \Rightarrow \log (A) = -1 \Rightarrow A = 10^{-1} = 0{,}1$**

**$G_{dB} = -40dB \Rightarrow 20 \log (A) = -40 \Rightarrow \log (A) = -2 \Rightarrow A = 10^{-2} = 0{,}01$**

**$G_{dB} = -3dB \Rightarrow 20 \log (A) = -3dB \Rightarrow \log (A) = -3/20 \Rightarrow A = 1/\sqrt{2}$**

**$G_{dB} = +6dB \Rightarrow 20 \log (A) = +6dB \Rightarrow \log (A) = 6/20 \Rightarrow A = 2$**

# 1.2. Bode plots

## 1.2.1. Characteristics of a Bode plot

In order to analyse the response of a linear quadrupole as a function of frequency, the **Bode plot** is the most practical representation.

A Bode plot is a set of two curves **as a function of frequency or angular frequency**:

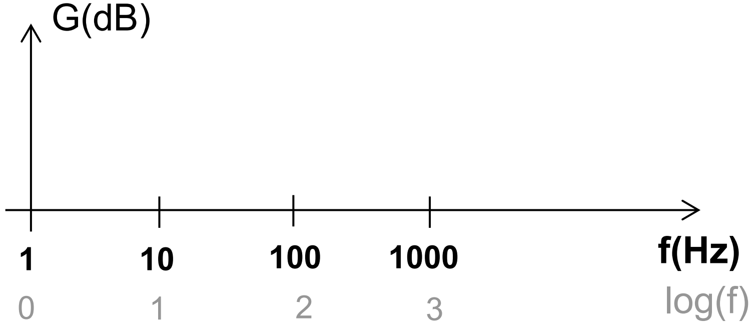
- The first one represents the **gain G**, in dB
- The second one represents the **phase shift ϕ**, in rad or degrees.

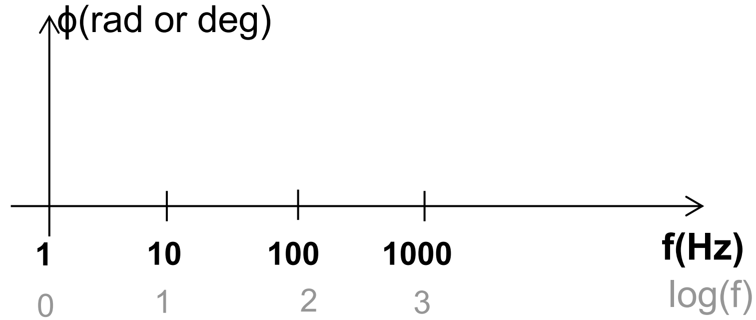
**Very important specificity of the Bode plot:**

a **logarithmic scale** is used for the **frequency or angular frequency**

*Why was a logarithmic scale chosen?*

A logarithmic scale allows a wide frequency range to be represented on a single graph. Each frequency decade has the same length on the scale (*e.g.* 1Hz to 10Hz has the same length as 100Hz to 1kHz). The horizontal axis is thus in fact **log (f), but only the values in Hz or Rad/sec are written.**



## A few definitions: decade, octave and choice of represented frequency range

Two frequencies $f_1$ and $f_2 > f_1$ are separated by a ***decade*** if $\mathbf{f_2 = 10 * f_1}$
Decades are already represented on sheets prepared for Bode plots

Two frequencies $f_1$ and $f_2 > f_1$ are separated by an ***octave*** if $\mathbf{f_2 = 2 * f_1}$

It is up to you to determine which values are represented on a Bode plot: you must choose them so that the most interesting parts of the plot is around the middle of the frequency axis.

In a Bode plot, the slopes of line segments are expressed either in **dB/decade**, or in **dB/octave.** For linear quadrupoles, the slopes are always multiples of 20 dB/decade (which practically corresponds to multiples of 6 dB/octave).

## ***Construction of a logarithmic scale***

A value of frequency f is chosen and placed according to its decimal logarithm **log(f),** f being expressed in Hz.

**-** *Construction of a decade:*

The following table is filled-in:

| **f** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
|---|---|---|---|---|---|---|---|---|---|---|
| **$\log_{10}$(f)** | **0** | **0,3** | **0,477** | **0,6** | **0,699** | **0,778** | **0,845** | **0,903** | **0,954** | **1** |

- Then the positions of $\log_{10}(x)$ are placed along the axis:


To cover several decades, all you need to do is to reproduce this scale (on the left and/or on the right) as many times as required. The frequency f is multiplied by 10 each time when moving one decade towards the right and divided by 10 each time when moving one decade towards the left.

**Comment**: for those who often need to trace Bode plots, specific sheets with a logartthmic horizontal scale have been designed (semi-logarithmic paper), avoiding the need to build the scale such as described above.

## 1.2.2. Asymptotic Bode plots of a few transfer functions.

The **asymptotic** Bode plot is an approximate plot that roughly describes the behaviour of a quadrupole using only straight line segments.

It is obtained by considering the behaviour at the frequency limits: f or $\omega \to 0$ and f or $\omega \to \infty$, considering that in a sum, only the dominant terms are taken into account.

Some particular values then appear in the transfer function.

We will determine asymptotic Bode plots of a few transfer functions.

*<u>A helpful comment for some transfer functions</u>: to obtain the Bode (or asymptotic Bode) plot of the* ***inverse*** *of a transfer function, you just need to change the sign of its Bode (or asymptotic Bode) plot.*

*<u>A helpful comment for complicated transfer functions</u>: the Bode (or asymptotic Bode) plot of the* ***product*** *of two transfer functions is the* ***sum*** *of each of their Bode (or asymptotic Bode) plots.*

**$T_1(j\omega) = j\omega/\omega_0$** ($\omega$ and $\omega_0$ are always positive)

$G_{dB} = 20 \log\omega - 20 \log\omega_0 = 20 \log\omega + cst$

$G_{dB}$ is a straight line with a slope of +20 dB/decade

$\phi = + \pi / 2$ rad or +90 deg

Here the asymptotic Bode plot and the Bode plot itself are identical.

**$T_2(j\omega) = 1 / j\omega/\omega_0 = 1 / T_1(j\omega)$**

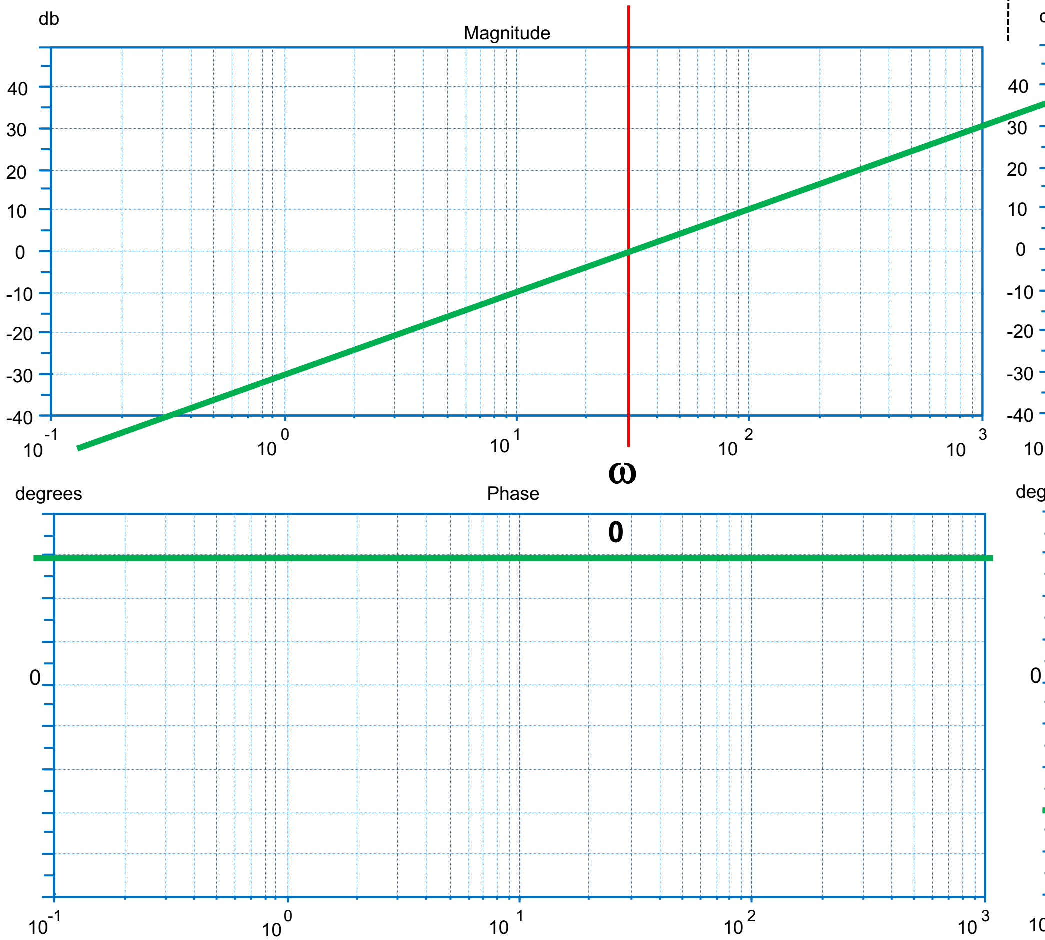
$G_{dB} = -20 \log\omega + 20 \log\omega_0 = -20 \log\omega + cst$

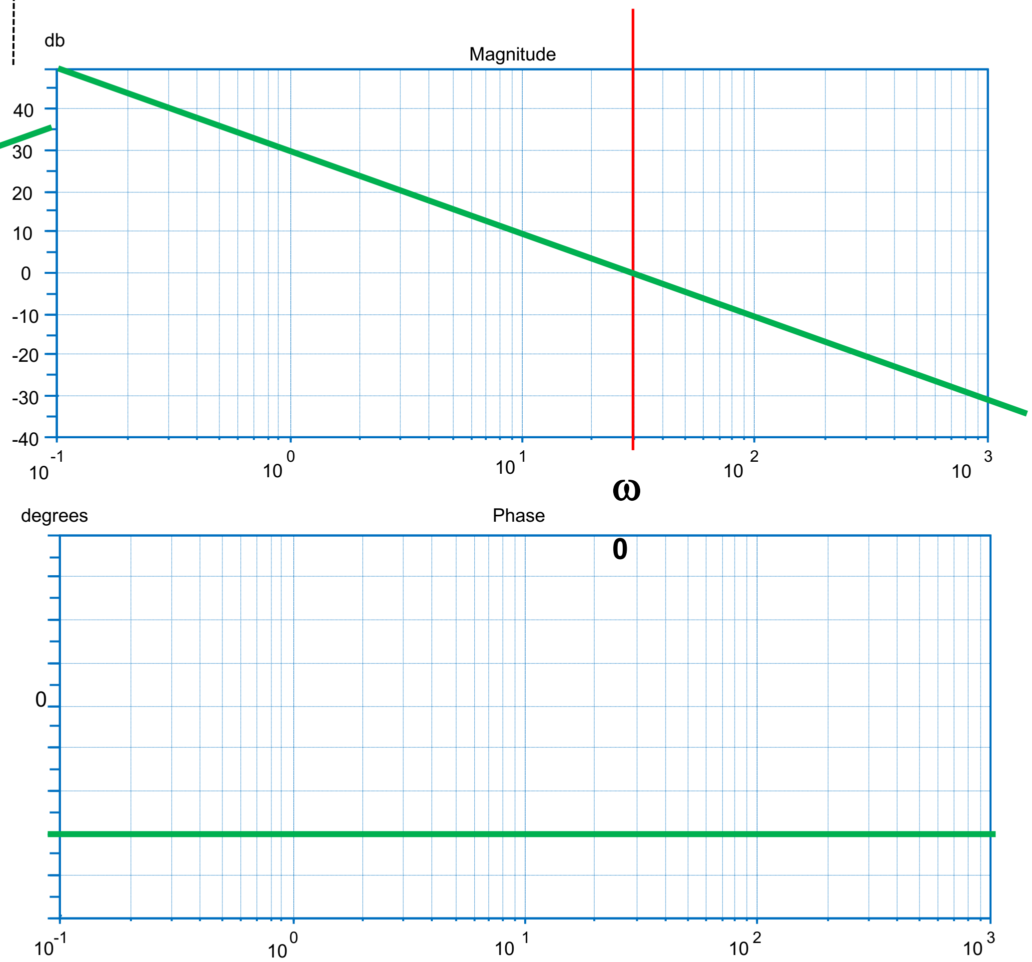
$G_{dB}$ is a straight line with a slope of -20 dB/decade

$\phi = - \pi / 2$ rad or -90 deg

The Bode plot of $T_2$ is minus the one of $T_1$

Here again the asymptotic Bode plot and the Bode plot itself are identical.

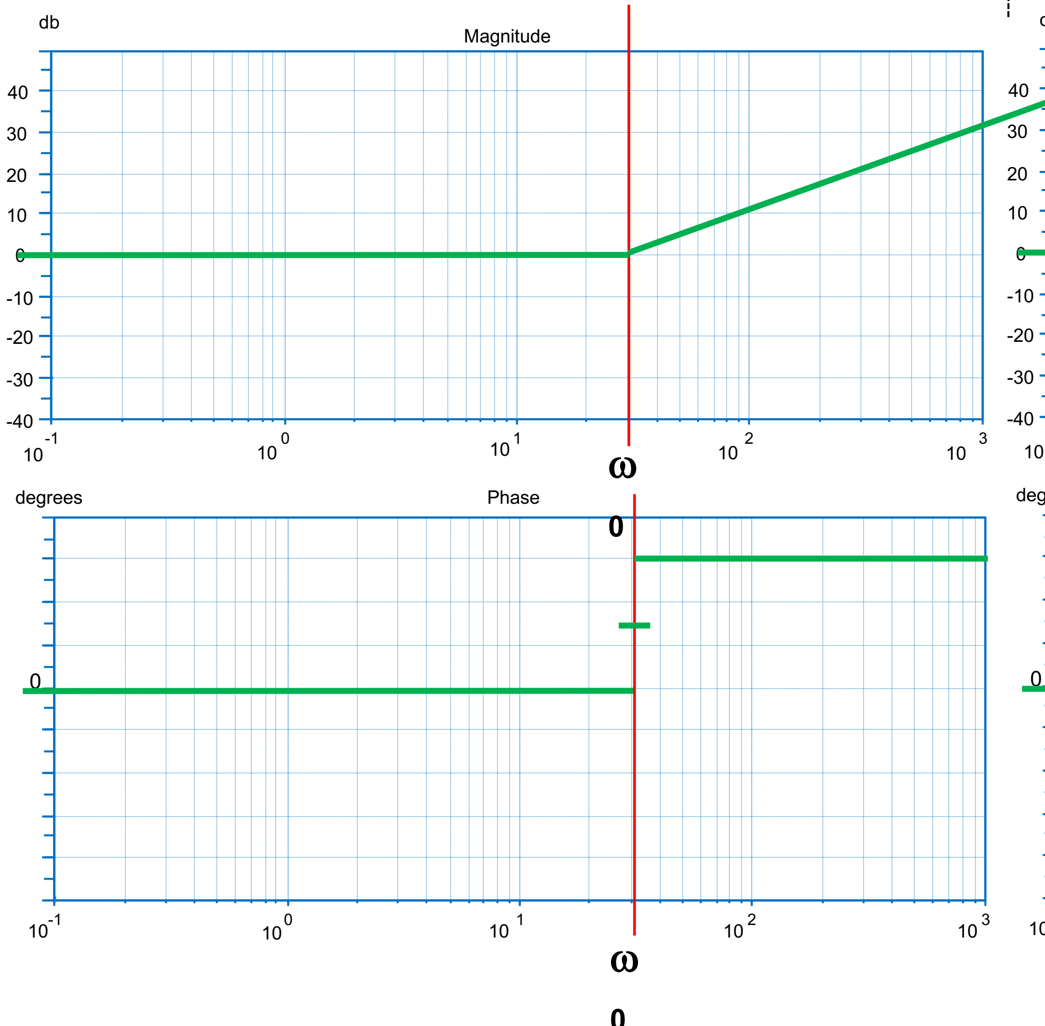


**$T_3(j\omega) = 1 + j\omega/\omega_0$**

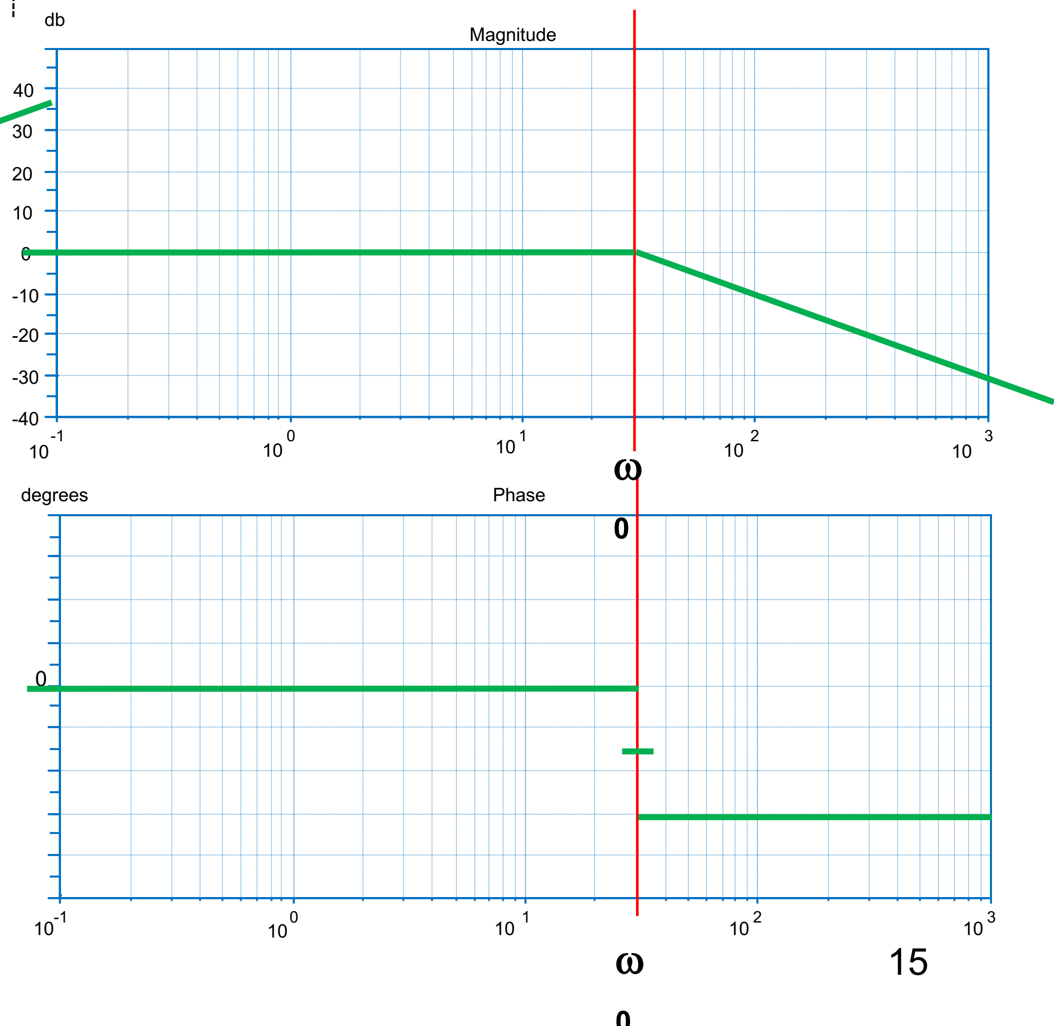
When $\omega \to 0$ $G_{dB} \to 20 \log 1 = 0$ and $\phi \to 0$

When $\omega \to \infty$ 1 becomes negligible compared to **$j\omega/\omega_0$**, $G_{dB} \to 20 \log\omega - 20 \log\omega_0 = 20 \log\omega + cst$ and $\phi \to + \pi / 2$ rad or +90 deg.




**$T_4(j\omega) = 1 / (1 + j\omega/\omega_0) = 1 / T_3(j\omega)$**

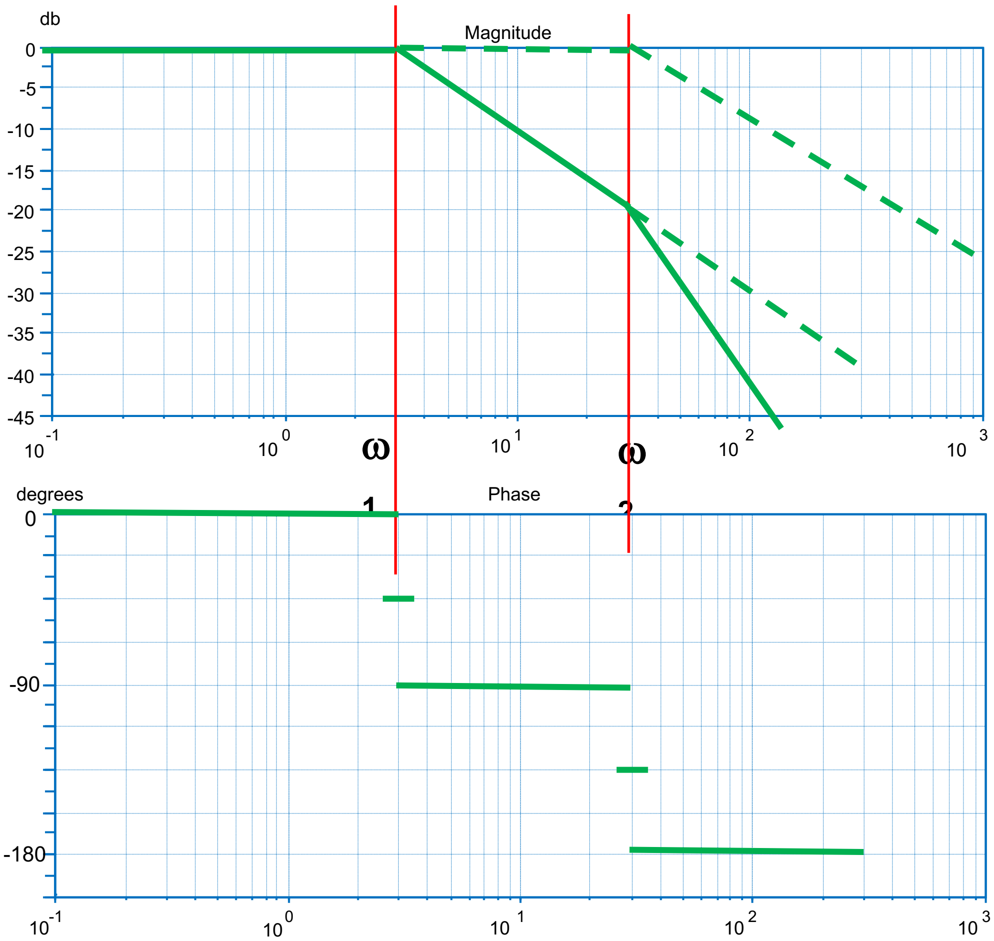
The asymptotic Bode plot of $T_4$ is minus the one of $T_3$ (so is the Bode plot itself).



$T_5(j\omega) = 1 \,/\, (1 + j\omega/\omega_1)(1 + j\omega/\omega_2) =$

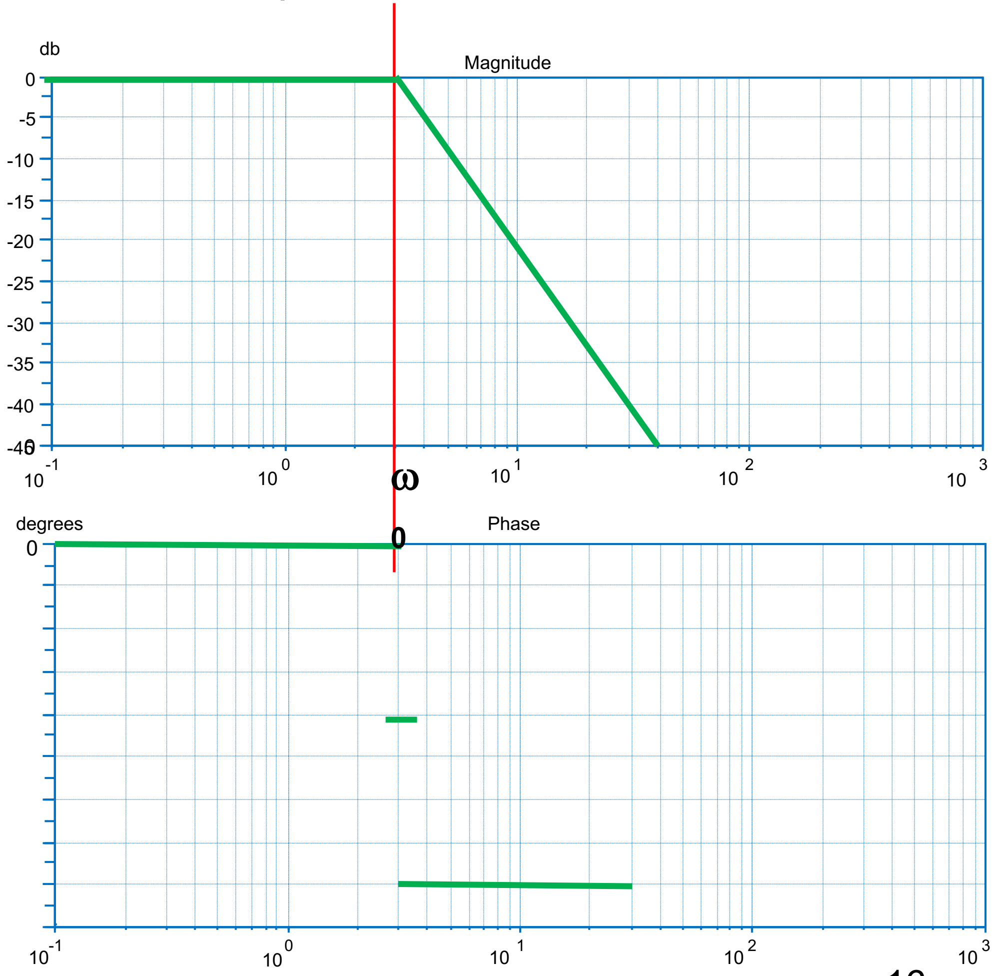
$1 \,/\, (1 + j\omega/\omega_1) \times 1 \,/\, (1 + j\omega/\omega_2)$

Let's use the results of $T_4$: dashed lines, first with $\omega_1$ second with $\omega_2$. **Then we just need to add the two asymptotic curves (solid line).** Slopes are (from left to right): 0, -20 and -40 dB/decade.




$T_6(j\omega) = 1 \,/\, (1+2mj\omega + (j\omega/\omega_0)^2)$,

If m > 1, $T_6$ can be expressed as $T_5$.
If m<=1, $T_6$ **cannot** be expressed as T5, then the asymptotic Bode plot is the one below below. Slopes are then 0 and -40 dB/decade.



# 1.3. First order filters

## 1.3.1. 1st order electrical circuits

In general, a transfer function can be written as a rational fraction of polynomial functions of jω:

$$\underline{T}(j\omega)=\frac{\underline{U}_s}{\underline{U}_e}=\frac{N(j\omega)}{D(j\omega)} \qquad \text{with} \quad \mathrm{Deg}\,(N(j\omega)) \le \mathrm{Deg}\,(D(j\omega))$$

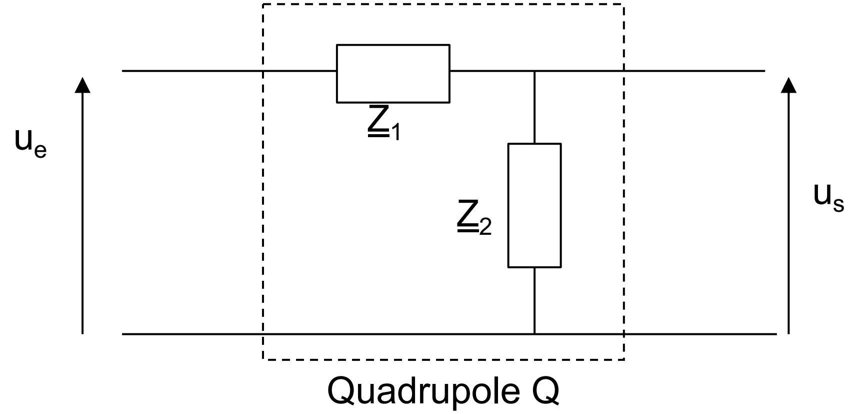
The order of the circuit is given by the degree of the polynomial function D(jω):

- if D(ω) is a 1st degree polynomial of jω, then the circuit is a 1st order one
- if D(ω) is a 2nd degree polynomial of jω, then the circuit is a 2nd order one
- etc.

Examples : $\underline{T}_1(\omega)=\dfrac{2}{1+j\cdot\dfrac{\omega}{6}}$ and $\underline{T}_2(f)=\dfrac{K}{1+j\cdot 2\pi fRC}$ are 1st order circuits

$\underline{T}_3(f)=\dfrac{1+3j\cdot f}{1+2j\cdot f+(300j\cdot f)^2}$ and $\underline{T}_4(\omega)=\dfrac{K'}{1+3j\cdot\omega RC+\left(j\dfrac{R}{L}\omega\right)^2}$ are 2nd order circuits

## 1.3.2 How to study simple quadrupoles / filters




Simple quadrupoles/filters can generally by represented as in the figure:

If the output current is zero, the transfer function can be obtained by applying the voltage divider formula:

$$\underline{H}(j\omega) = U_s / U_e = Z_2 / (Z_1 + Z_2)$$

***Magnitude:***

***$A(\omega)$ and $G_{dB}(\omega)$***

***Phase shift:***

***$\varphi(\omega)$***

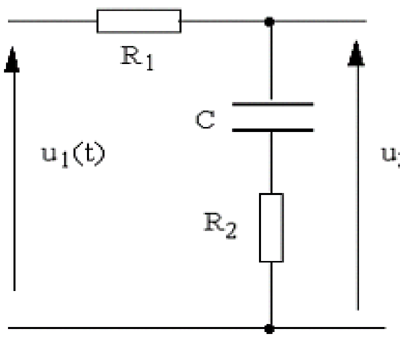
Variations of the magnitude and phase shift as functions of angular frequency can then be studied separately.

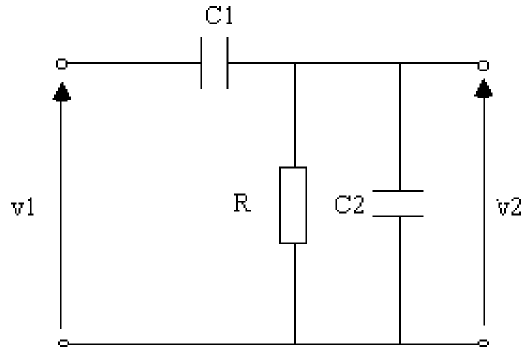
Often, **an asymptotic** study is sufficient:

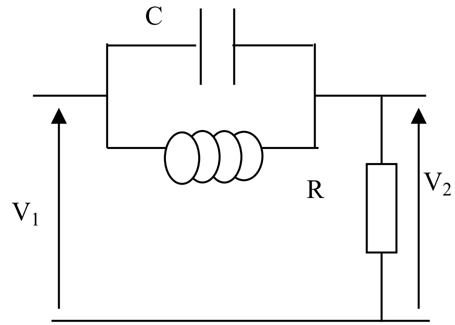
- Low frequency behaviour ($\omega \to 0$)
- High frequency behaviour ($\omega \to +\infty$)
- Specific values: maximum,…

***Remark:***

Most often, the order of a circuit can be determined by the number of capacitors and inductors that it contains. Only 1 inductor or capacitor: 1st order circuit.



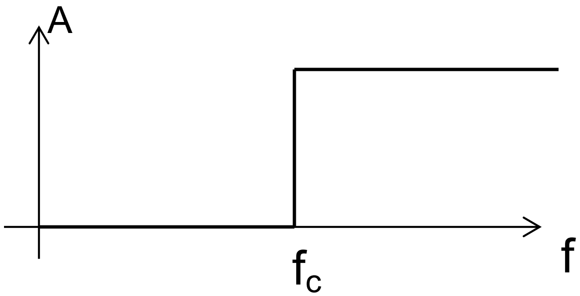

### 1.3.2. Filtering circuits (or filters)

A filter is an electrical circuit that contains elements assembled in a way to selectively transmit signals in a given frequency range. An ideal filter has one or several pass-bands in which signals are transmitted without being attenuated and one or several stop-bands in which signals are attenuated or stopped.

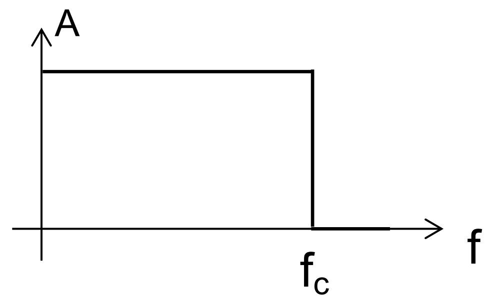
Filters can be characterised by:

- Their amplification or atténuation
- The frequency bands in which they operate: low-pass, high-pass, band-pass notch
- Their technology: active filters (can amplify signals because they contain at least one active component) of passive filter (made only of capacitors, inductors and resistors).

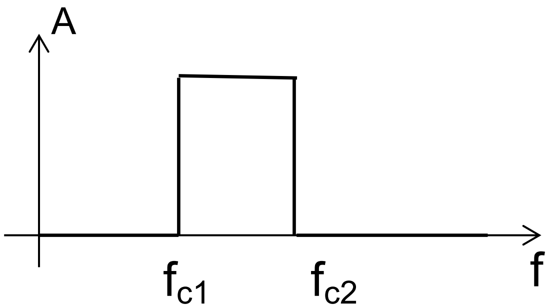
## ***Ideal filters***



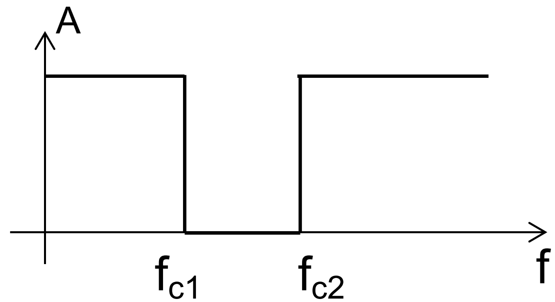

High-pass filter




Low-pass filter




Bandpass filter




Notch filter

***Definition** of the cut-off frequency(ies) of a filter.*

Cut-off frequencies are the frequency limits of a filter's passband, they are defined by the following condition on the magnitudes:

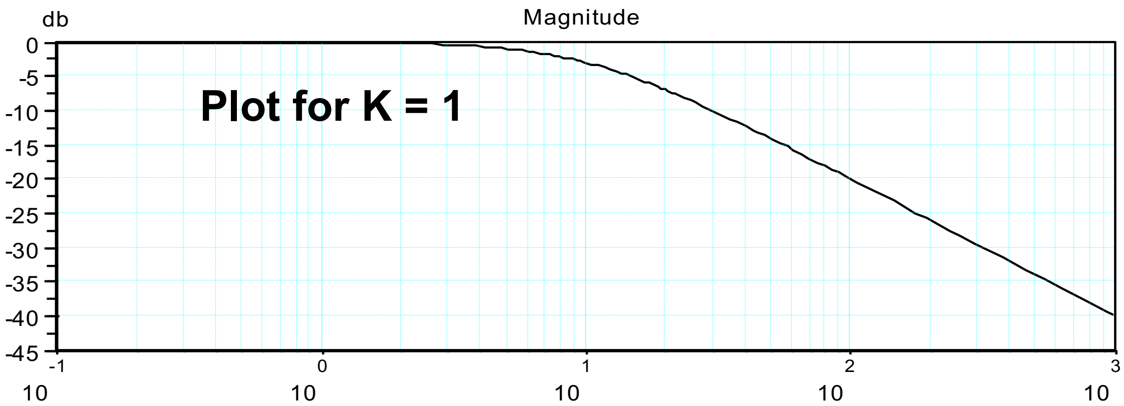
$$\mathbf{G_{dB}\,(f_c) = G_{dB\,max} - 3\ dB \quad \Leftrightarrow \quad A(f_c) = A_{max} / \sqrt{2}}$$

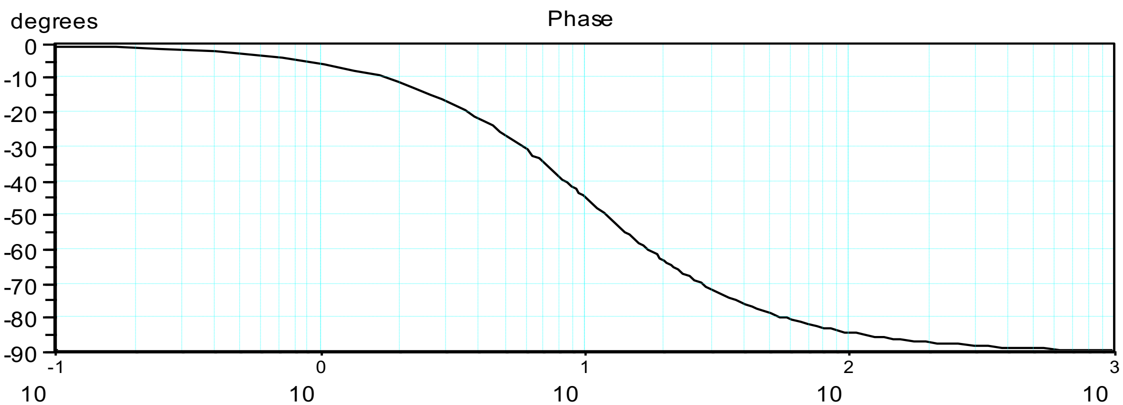
## 1.3.3. Bode plots of first order filters

A first order filter is a *1st order electrical circuit* that allows input signals to be attenuated (cut or stopped) in a certain range of frequencies. In this stopband, the magnitude curve of the Bode plot of a first order filter always has a ±*20 dB/decade* slope (or ±*6 dB/octave)*.

*Low-pass filter:* its transfer function is given by

$$\underline{T}(j\omega) = \frac{K}{1 + j \cdot \frac{\omega}{\omega_c}}$$




A low-pass filter:
- does not attenuate low frequency signals, *i.e.* between 0 Hz et $f_c$.
- attenuates (filters out) high frequency signals, *i.e.* between $f_c$ and+∞.

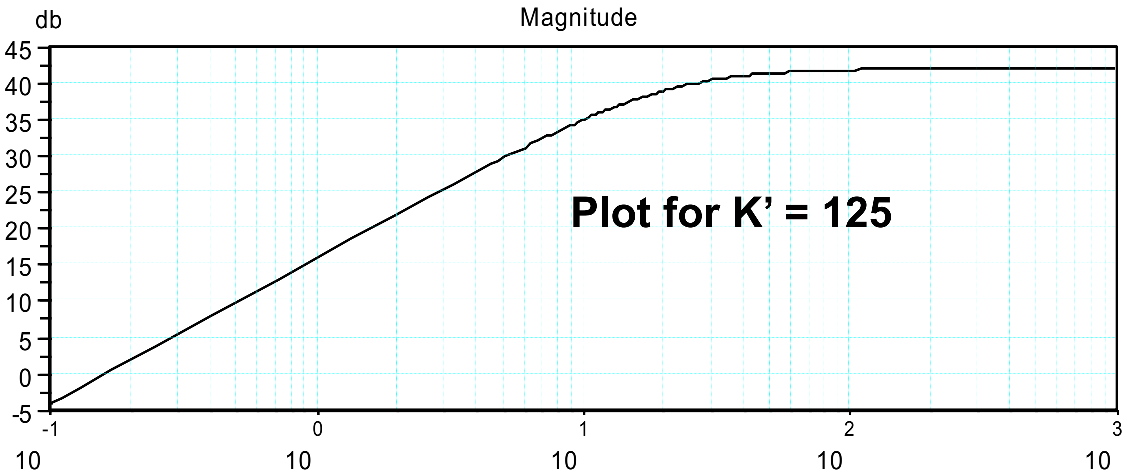
-3dB bandpass: $[0 ; f_c]$

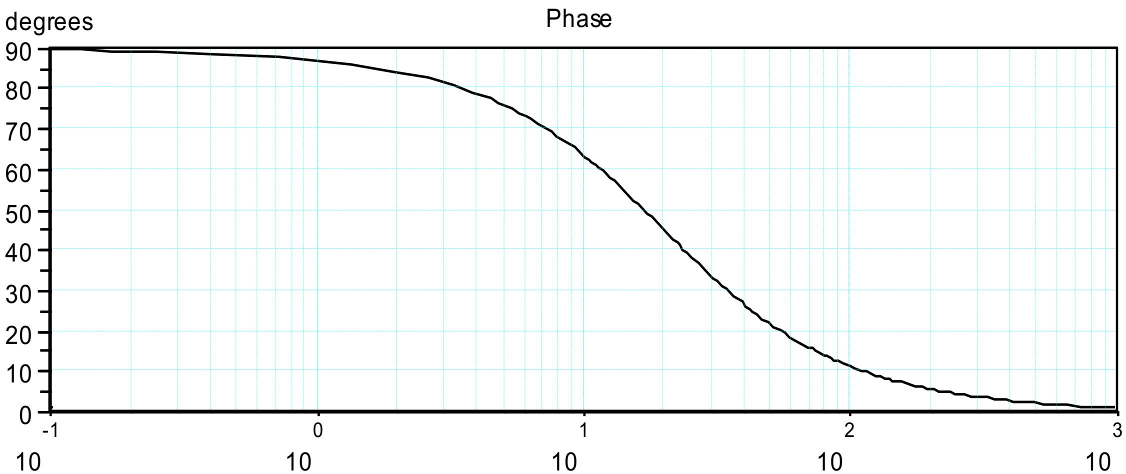
stop-band: $]f_c ; +\infty[$

$f_c$: cut-off frequency

*<u>High-pass filter:</u>* its transfer function is given by

$$\underline{T}(f) = K' \cdot \frac{j \cdot \dfrac{f}{f_c}}{1 + j \cdot \dfrac{f}{f_c}}$$




A high-pass filter:

- does not attenuate high frequency signals, *i.e.* between $f_c$ and $+\infty$.
- attenuates (filters out) low frequency signals *i.e.* between 0 Hz and $f_c$.

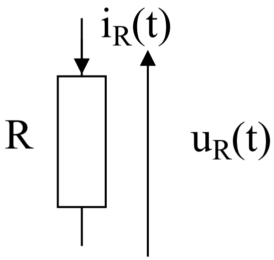
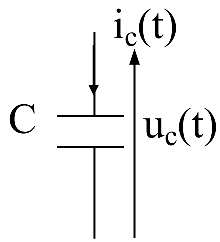
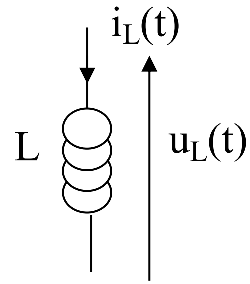
-3dB bandpass: $]f_c ; +\infty[$

stop-band: $[0 ; f_c]$

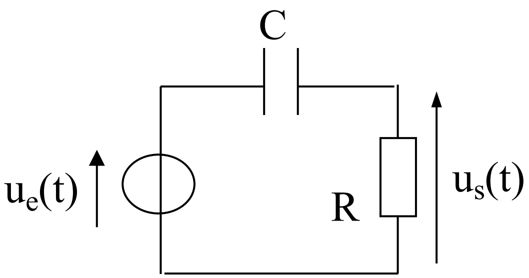
$f_c$: cut-off frequency

## 1.3.4 Qualitative analysis of filters

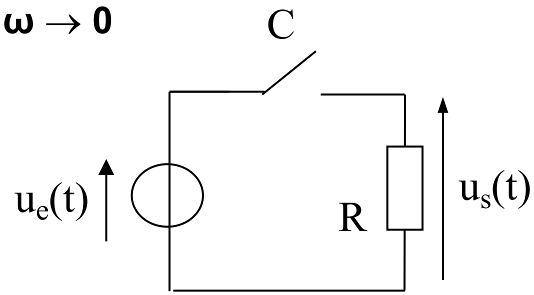
The qualitative analysis of a filter consists in the determination of its behaviour for extreme frequencies, *i.e.* f → 0 and f → +∞. For this one needs only to replace the dipoles by their equivalent at these frequency limits:

| | | | |
|---|---|---|---|
| | <br>$\underline{Z}_R = R$ | <br>$\underline{Z}_C = 1/jC\omega$ | <br>$\underline{Z}_L = jL\omega$ |
| **$\omega \to 0$** | $Z_R = R$ | $Z_C \to \infty$<br>Capacitor equivalent to an open circuit | $Z_L \to 0$<br>Inductor equivalent to a short circuit |
| **$\omega \to +\infty$** | $Z_R = R$ | $Z_C \to 0$<br>Capacitor equivalent to a short circuit | $Z_L \to \infty$<br>Inductor equivalent to an open circuit |

## 1.3.4 Qualitative analysis of first order filters

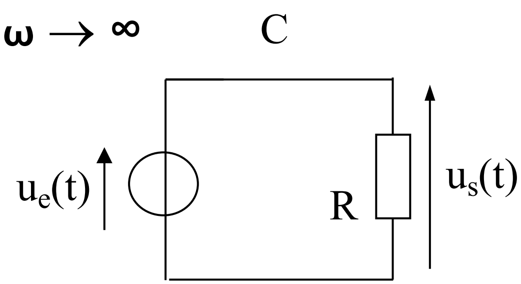



**ω → 0**



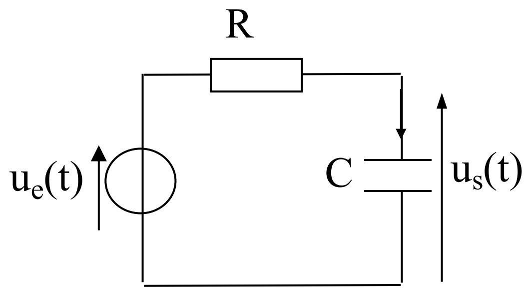

**$u_s(t) = 0$**

**ω → ∞**

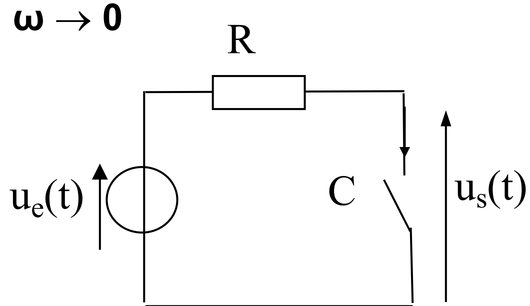



**$u_s(t) = u_e(t)$**

=> High-pass filter

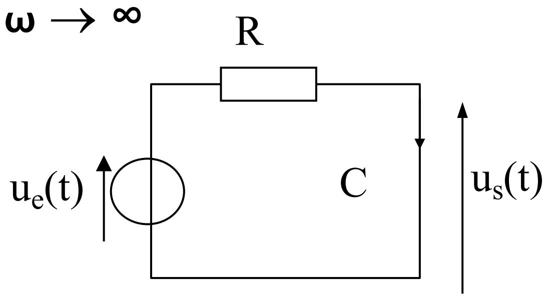



**ω → 0**



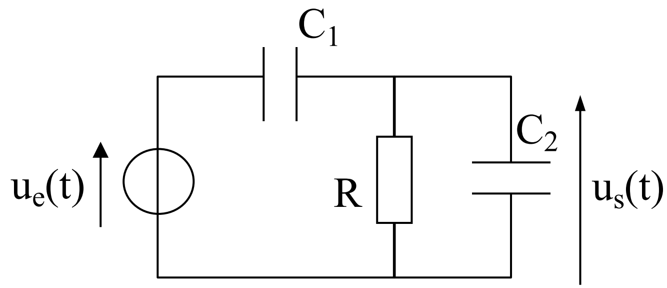

**$u_s(t) = u_e(t)$**

**ω → ∞**

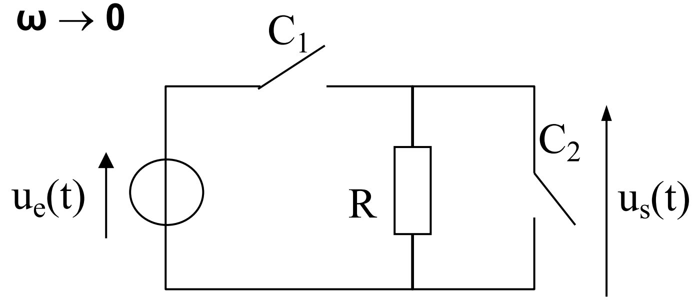



**$u_s(t) = 0$**

=> Low-pass filter

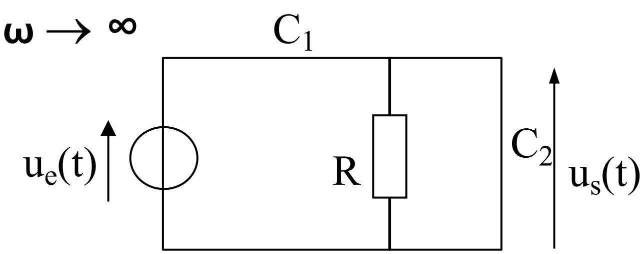



**ω → 0**




**$u_s(t) = 0$**

**ω → ∞**




**$u_s(t) = 0$**

=> Band-pass filter

# 1.4. Second order filters

The 4 types of typical 2nd order filters are characterised by the following transfer functions:

Low-pass (or high-cut) filters:

$$\underline{T}_{bas}(\omega) = K \frac{1}{1 + 2m.j\frac{\omega}{\omega_0} + \left(j\frac{\omega}{\omega_0}\right)^2}$$

High-pass (or low-cut) filters:

$$\underline{T}_{haut}(\omega) = K \frac{\left(j\frac{\omega}{\omega_0}\right)^2}{1 + 2m.j\frac{\omega}{\omega_0} + \left(j\frac{\omega}{\omega_0}\right)^2}$$

Bandpass filters:

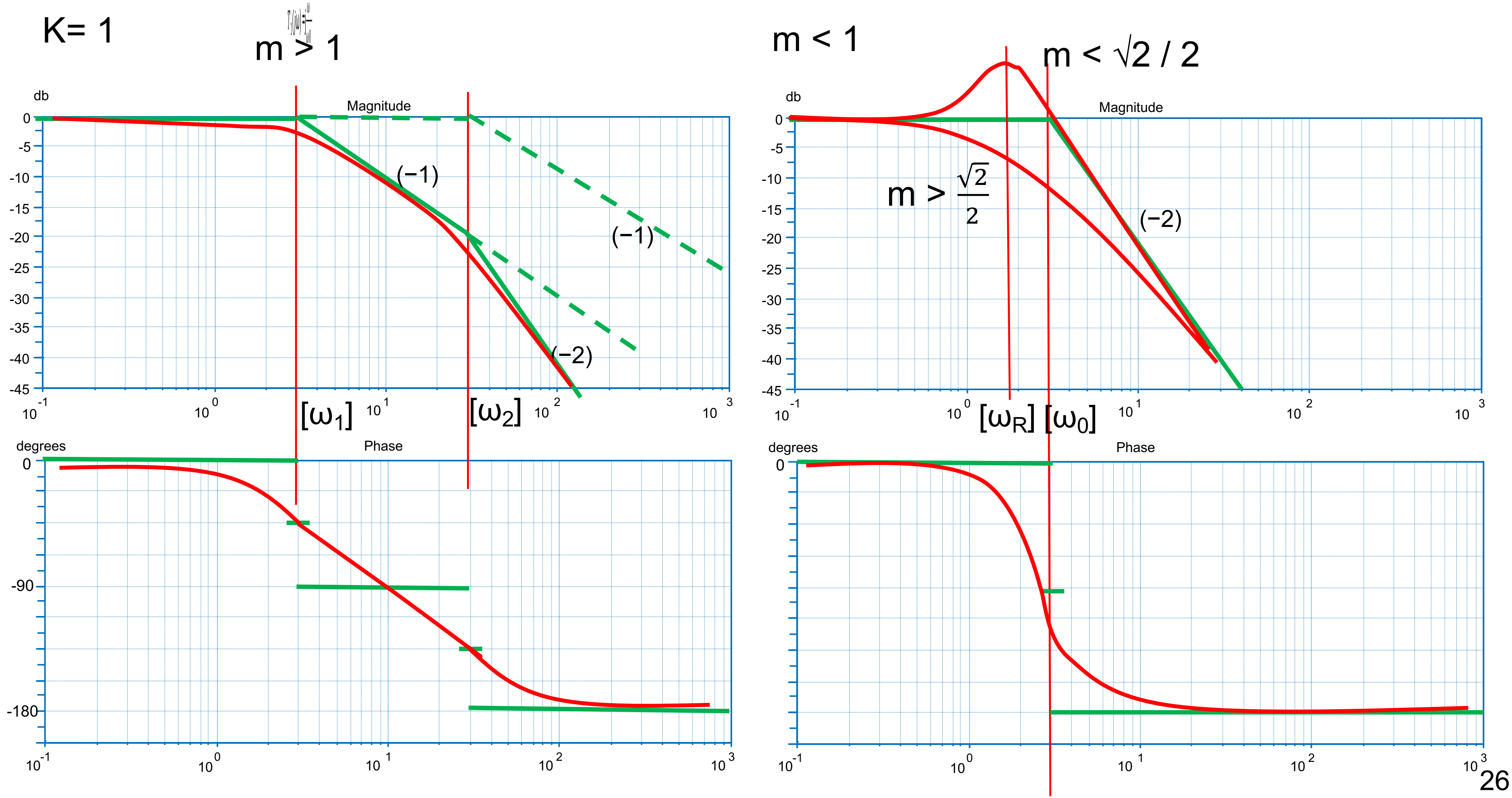
$$\underline{T}_{bande}(\omega) = K \frac{2m.j\frac{\omega}{\omega_0}}{1 + 2m.j\frac{\omega}{\omega_0} + \left(j\frac{\omega}{\omega_0}\right)^2}$$

Notch filters:

$$\underline{T}_{réj}(\omega) = K \frac{1 + \left(j\frac{\omega}{\omega_0}\right)^2}{1 + 2m.j\frac{\omega}{\omega_0} + \left(j\frac{\omega}{\omega_0}\right)^2}$$

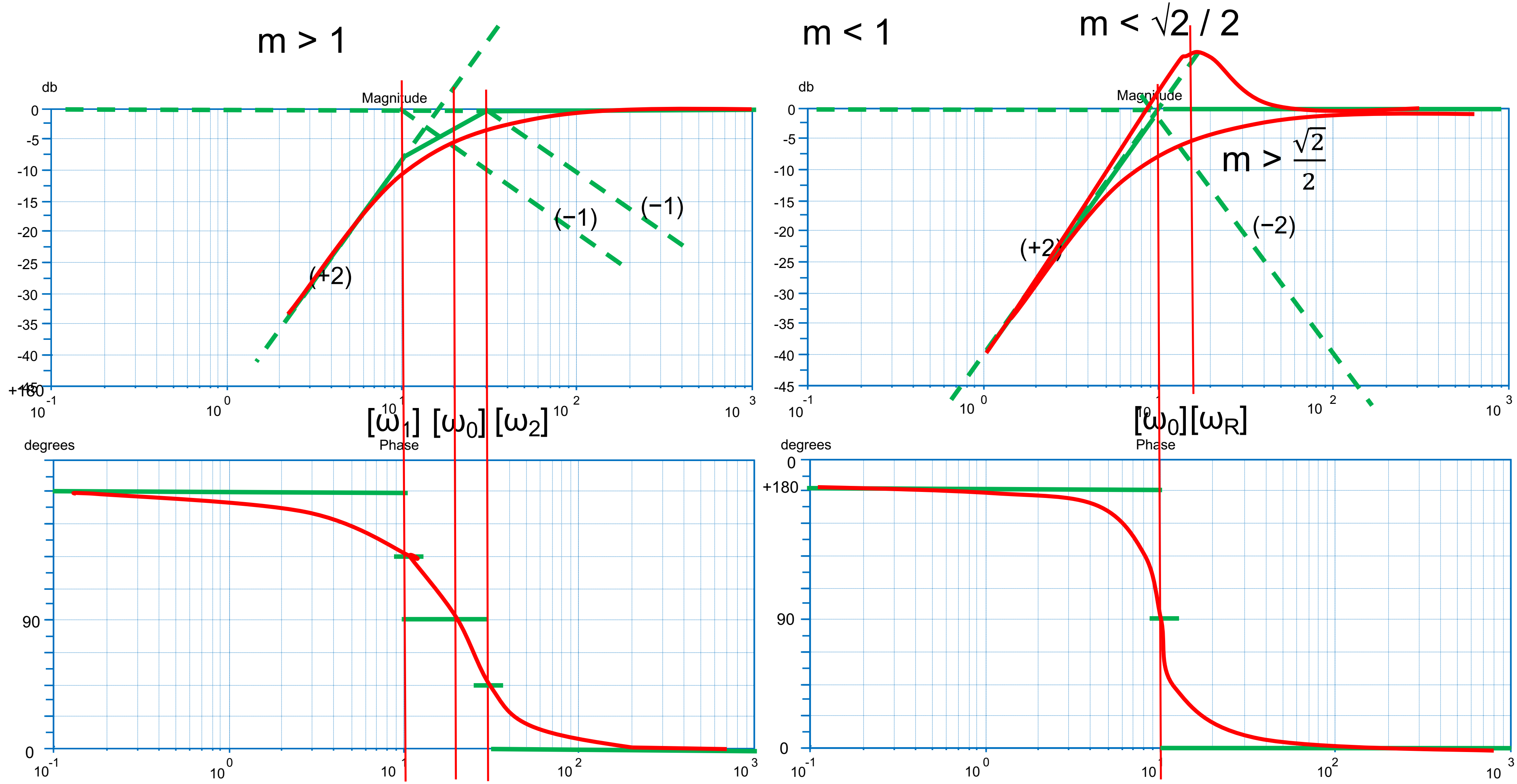
## 1.4.1. Bode plots of second order filter

① Low-pass filters: $$\underline{T}_{bas}(\omega) = K \frac{1}{1 + 2m.j\dfrac{\omega}{\omega_0} + \left(j\dfrac{\omega}{\omega_0}\right)^2}$$



② High-pass (or low-cut) filters: $\underline{T}_{haut}(\omega) = K \dfrac{\left(j\dfrac{\omega}{\omega_0}\right)^2}{1 + 2m.j\dfrac{\omega}{\omega_0} + \left(j\dfrac{\omega}{\omega_0}\right)^2}$

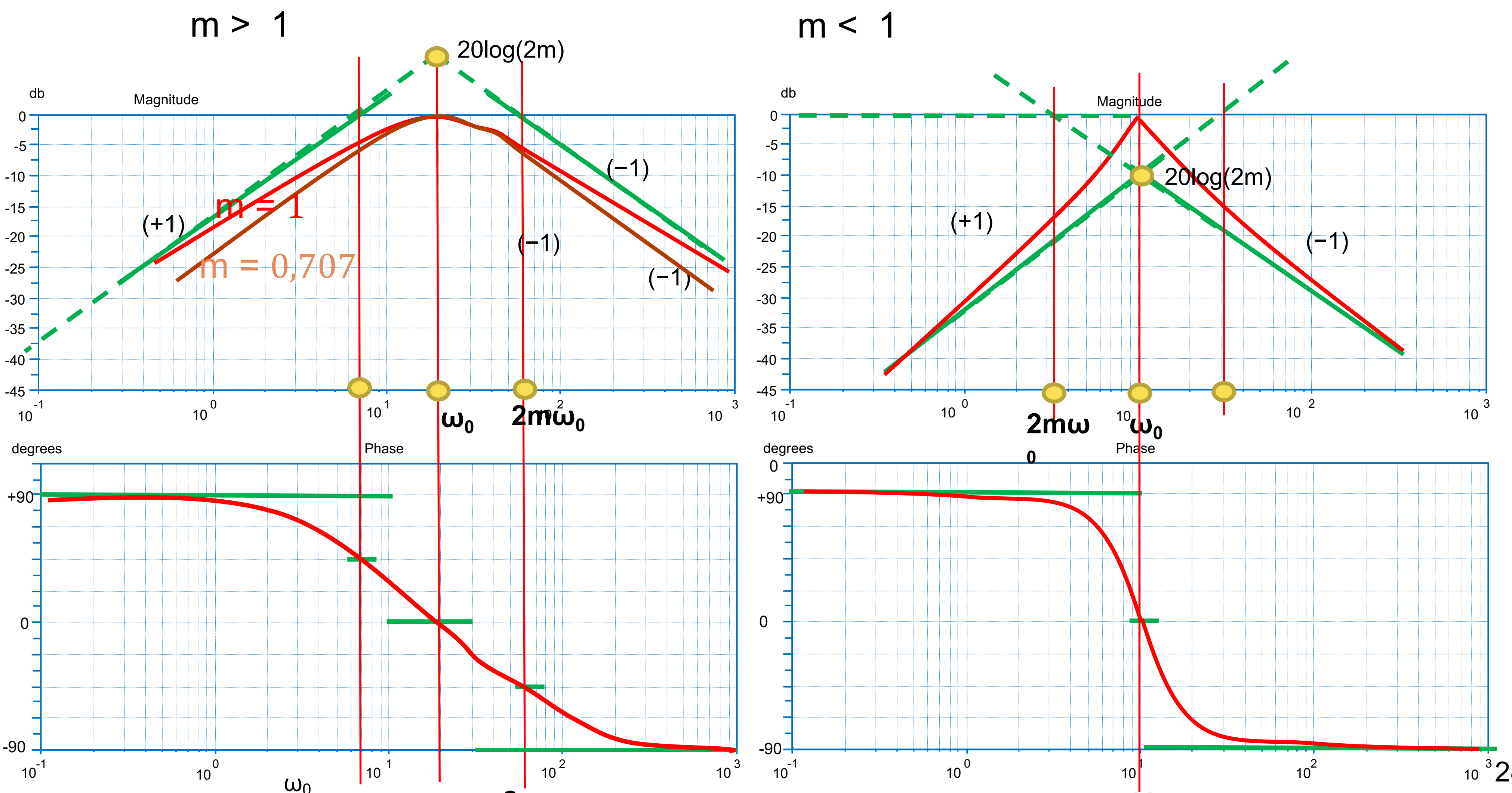
K= 1



③ Bandpass filters:

$$\underline{T}_{bande}(\omega) = K \frac{2m.j\dfrac{\omega}{\omega_0}}{1 + 2m.j\dfrac{\omega}{\omega_0} + \left(j\dfrac{\omega}{\omega_0}\right)^2}$$

K= 1



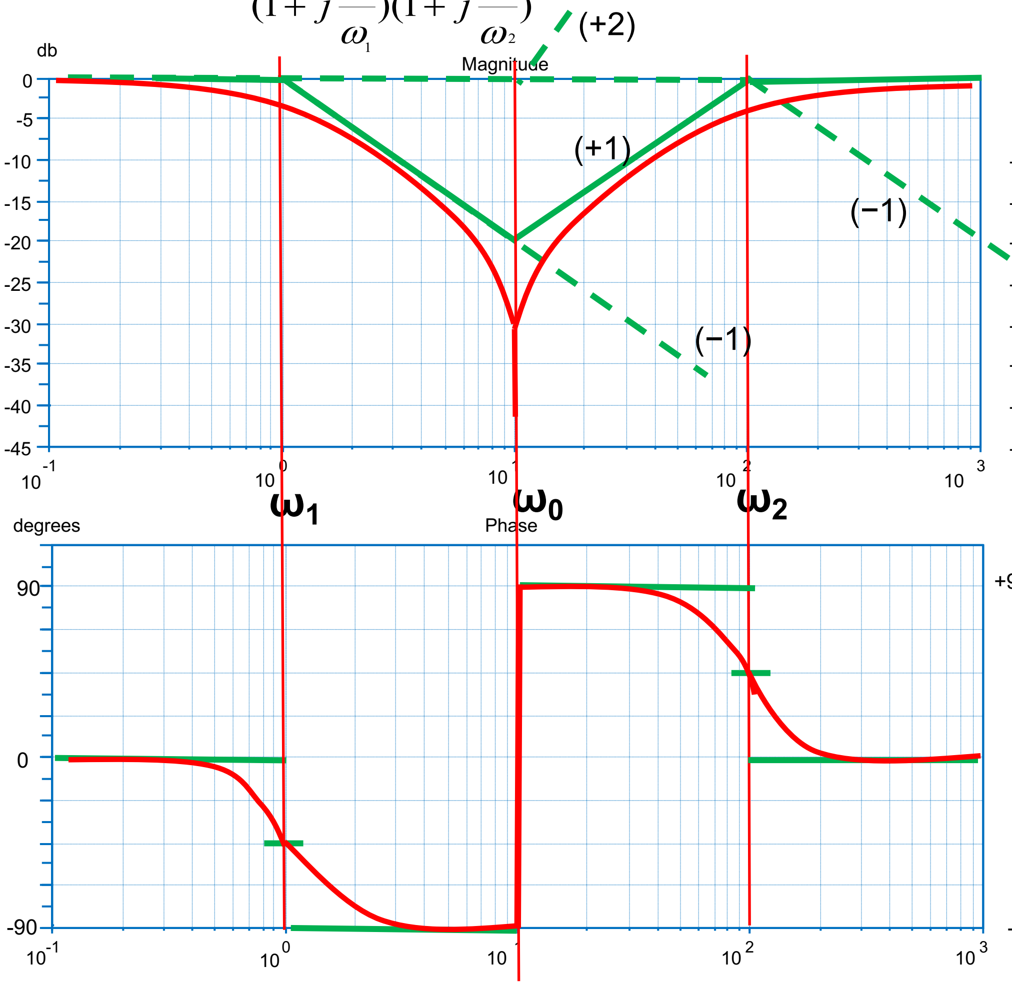
## ④ Notch filters:

$$\underline{T}_{réj}(\omega) = K\frac{1+\left(j\frac{\omega}{\omega_0}\right)^2}{1+2m.j\frac{\omega}{\omega_0}+\left(j\frac{\omega}{\omega_0}\right)^2} = \frac{\underline{N}(\omega)}{\underline{D}(\omega)}$$

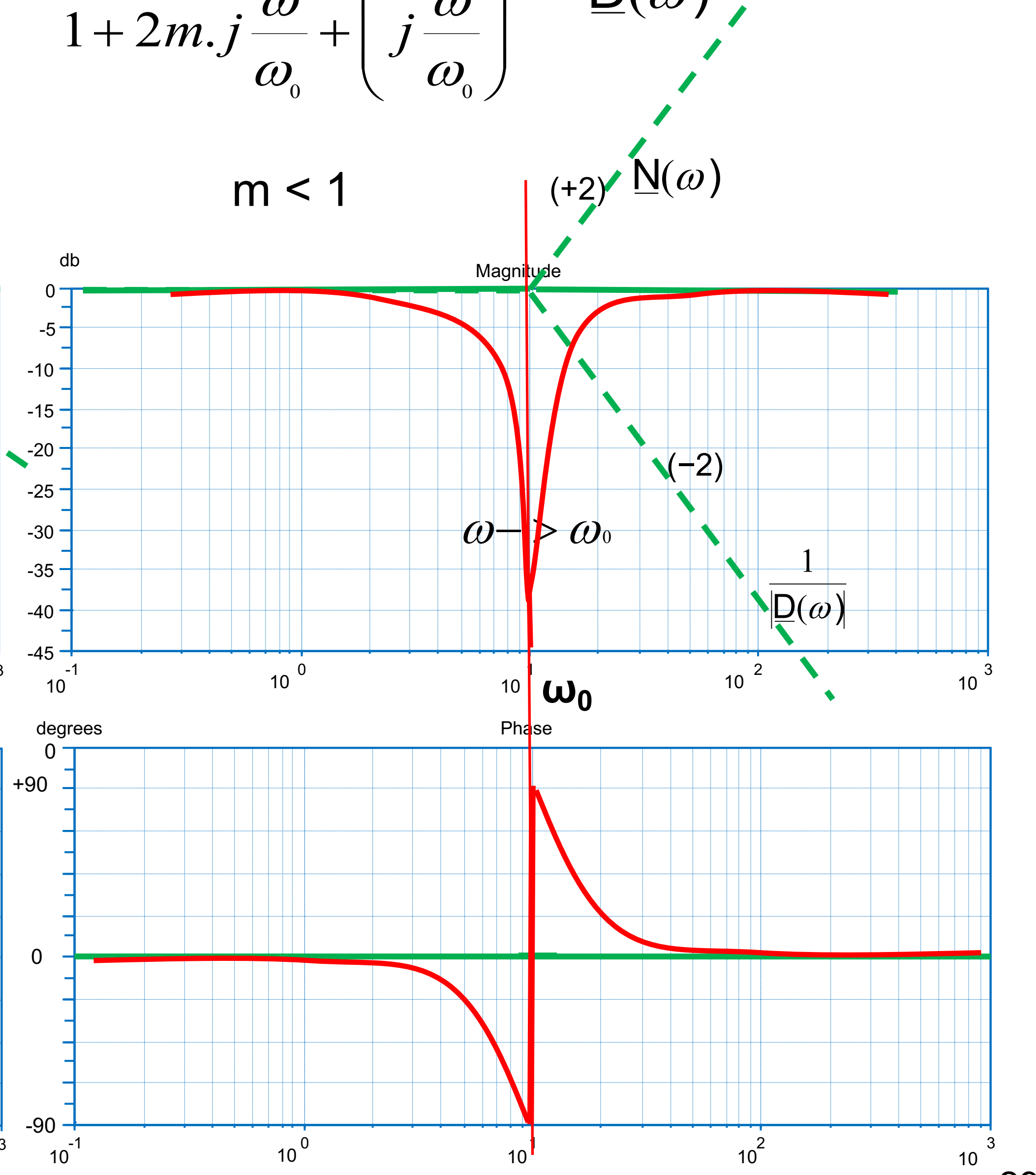
K= 1

m > 1 :

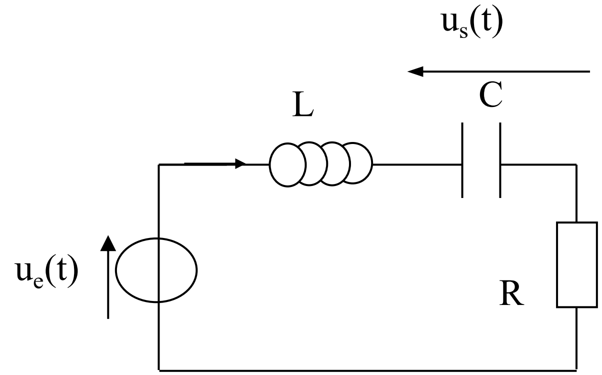
$$\underline{T}_{réj}(\omega) = K\frac{1+\left(j\frac{\omega}{\omega_0}\right)^2}{(1+j\frac{\omega}{\omega_1})(1+j\frac{\omega}{\omega_2})}$$

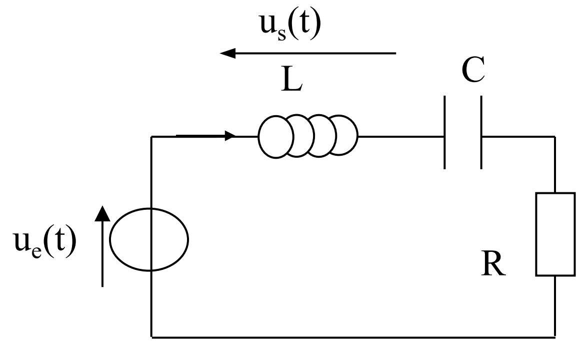


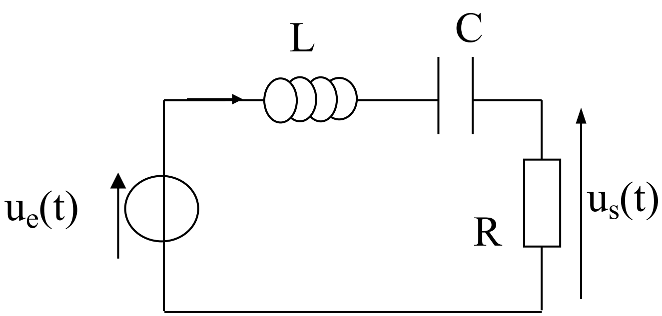

m < 1

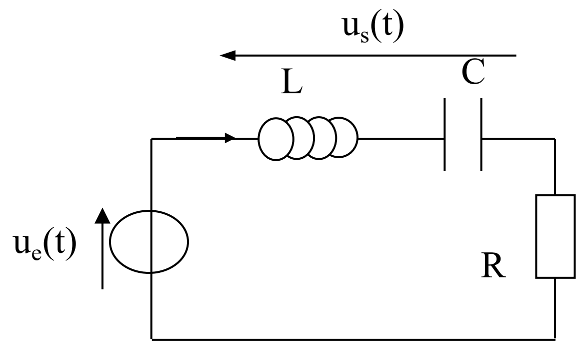


## 1.4.2. Qualitative analysis of second order filters



## 1.4.2. Qualitative analysis of second order filters