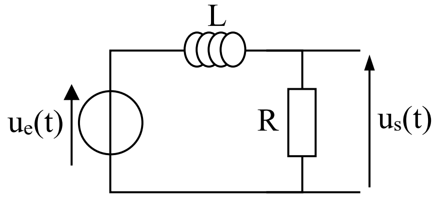
Exercises on filters

I 1st order filters

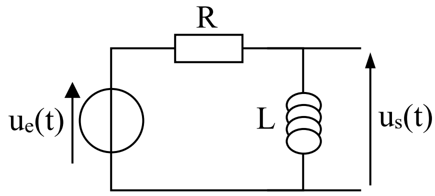
A) LR circuit



Let us consider a circuit made of an inductor L and a resistor R (figure on the right); a sinusoidal voltage of angular frequency ω, noted $u_e(t)$, is applied at its input.
To obtain its Bode plot, the following values are given: R= 1kΩ and L=5mH.

1) Determine this circuit's transfer function $\underline{H}(j\omega) = \underline{U}_s/\underline{U}_e$.
2) Write the expression of its gain in decibels $|\underline{H}|_{dB}$; then determine its values at the limit frequencies, *i.e.* for $\omega \to 0$ and $\omega \to \infty$.
3) From these results, give the gain Bode plot of this quadrupole.
4) Answer questions **2)** and **3)** for the phase shift of $\underline{H}$.
5) Knowing that this circuit is going to be used as a filter, indicate which frequencies will be attenuated. What can be said about the phase shift?

B) RL circuit



Same questions if the positions of R and are switched:

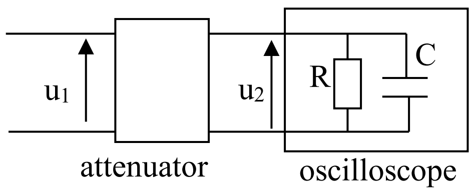
Comment: in practice, particularly for low-power applications, an inductor of 5mH will not be used because of its big size. A filter made of a capacitor and a resistance will be preferred.

C) Circuits using a capacitor C

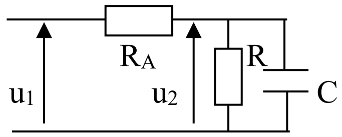
1) Draw a circuit with the same transfer function as in A using a capacitor and a resistor, then determine the value of C if R= 1kΩ.
2) Draw a circuit with the same transfer function as in B using a capacitor and a resistor, then determine the value of C if R= 1kΩ.

II Principle of an oscilloscope's attenuator

A classical oscilloscope allows visualising voltages up to 40V peak-to-peak. For higher voltages, one can use an attenuator connected to the oscilloscope's input in order to divide the voltage by a given coefficient.
The input of an oscilloscope, seen from its external connections, is equivalent to a resistor R in parallel with a capacitor C. If u_1 is the voltage to be measured, the voltage at the input of the oscilloscope will be u_2 such as: $\frac{U_2}{U_1} = \alpha$ (in all this study, the sinusoidal voltage $u_1(t)$ is characterised by an angular frequency ω and an amplitude U_1, the amplitude of $u_2(t)$ is U_2):



1) The most simple solution one can think of consists in adding a **resistor** R_A connected as in the figure below:



a) Show that the transfer function $H(j\omega) = \dfrac{\underline{u}_2}{\underline{u}_1}$ can be written as $H(j\omega) = \dfrac{\alpha}{1 + j\dfrac{\omega}{\omega_0}}$. Find α and ω_0 as as functions of R, C and R_A.

b) What type of filter is this? Determine it cut-off angular frequency ω_C.

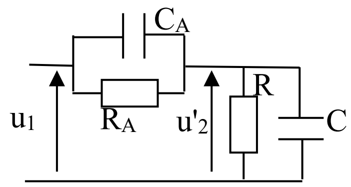
c) Determine the equations of the asymptotes of the gain curve of the Bode plot of this filter, in particular give their intersection and slopes.

d) Draw the asymptotic Bode plot of the gain curve on the semi-logarithmic sheet provided while carefully graduating the axis.

e) What happens if the frequency of the measured voltage varies? What is the defect of this attenuator?

f) Give the physical interpretation of the role of the capacitor.

2) In order to avoid this defect, a capacitor C_A connected in parallel R_A is added as shown in the circuit opposite:



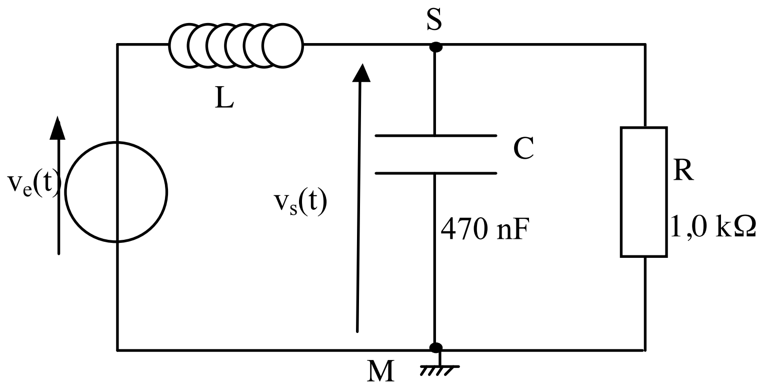
a) Determine the new transfer function $H'(j\omega) = \dfrac{\underline{u}_2'}{\underline{u}_1}$, then white is as $H'(j\omega) = \alpha.\dfrac{1 + j\dfrac{\omega}{\omega_1}}{1 + j\dfrac{\omega}{\omega_2}}$. Find α, ω_1 and ω_2 as functions of R, C, R_A and C_A.

b) What is the relation between R, C, R_A and C_A that leads to a constant transfer function? Then, write *H'(jω)*.

c) Numerical calculation: a classical oscilloscope is characterised by R = 1,0 MΩ and C = 20 pF. Calculate the values that should be given to R_A and C_A in order to obtain an attenuation of $\alpha = \dfrac{1}{10}$ whatever the frequency.

III Study of a 2nd order low-pass filter

Circuit 1



III.1/ Time-domain analysis

$v_e(t)$ being a step voltage with a 5V amplitude, with an initial value of 0V at t = 0:

1) Give the differential equation linking the output voltage $v_s(t)$ to $v_e(t)$. What are the three possible expressions of $v_s(t)$ according to the values of the inductor L?

2) Draw representations of $v_s(t)$ for inductor values of 100 mH, of 10 H and for the critical value L_c.

III.2/ Frequency-domain analysis

3) Find the expression of the transfer function and put it is the form $T(j\omega) = \dfrac{K}{1 + 2j\xi\dfrac{\omega}{\omega_0} + \left(j\dfrac{\omega}{\omega_0}\right)^2}$.

4) We wish to obtain the corresponding Bode plot.

- **a)** For this, factorise the denominator of T(jω); show that this is possible only for certain values of ξ; then give the limit value of L.
- **b)** In the case where the factorisation is possible, draw the asymptotic Bode plot, then the Bode plot itself for a given value of the inductor.
- **c)** In the case where the factorisation is NOT possible, what can you notice (refer to TP n°4 of EC1)?

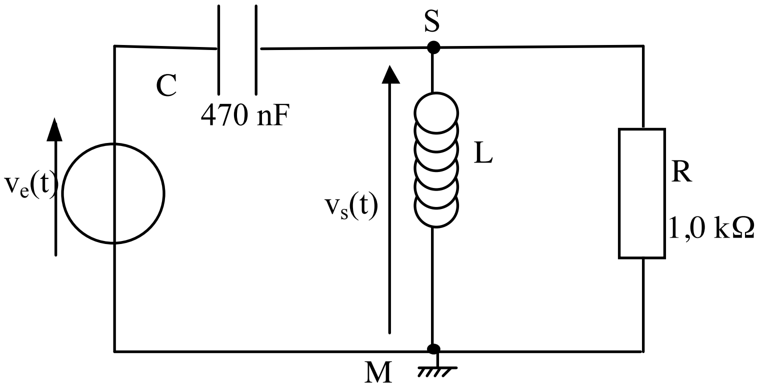
5) We will now study more precisely the case where the factorisation is not possible.

- **a)** Study the variations of the denominator.
- **b)** In which cases can a resonance be observed?
- **c)** Give the expressions of the resonance angular frequency ω_R, as well as the maximum of T(jω)'s magnitude and the corresponding gain.
- **d)** Find the expression of the cut-off angular frequency ω_C of this filter.
- **e)** Determine the magnitude and phase shift of T(jω) at $\omega = \omega_0$.
- **f)** Draw the asymptotic Bode plot, then the Bode plot itself, for two values of inductor L : one for which there is no resonance and one for which there is a resonance.

IV Study of a 2nd order high-pass filter

We will now study the circuit in which the positions of the inductor and capacitor have been switched.

Circuit 2



1) By a simple qualitative analysis, check that this is indeed a high-pass filter.

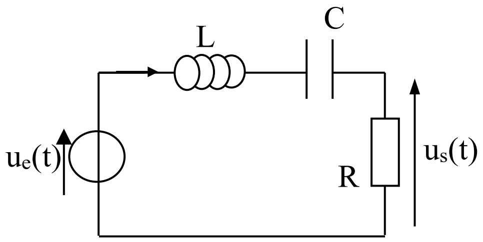
2) Find its transfer function and express it as $T(j\omega) = \dfrac{K\left(j\dfrac{\omega}{\omega_0}\right)^2}{1 + 2j\xi\dfrac{\omega}{\omega_0} + \left(j\dfrac{\omega}{\omega_0}\right)^2}$.

3) Show that the conditions found in questions **4) a)** and **5) b)** of exercise **III** also apply here.
4) Carefully draw the asymptotic Bode plot, then the Bode plot itself for the same values of the inductor L as in the previous exercise (use the same approach as in exercise **III**).

V Series R, L, C circuit

We want to study the evolution of the voltage across the resistor R of a series R, L, C circuit (figure below).

R = 470 Ω
L = 220 mH
C = 47 nF



1) By a simple qualitative analysis, determine which is the type of this filter.

2) Find the transfer function of this filter and show that it can be written in the following canonical form:

$$\underline{T}(j\omega)=K\frac{2jm\frac{\omega}{\omega_0}}{1+2jm\frac{\omega}{\omega_0}+\left(j\frac{\omega}{\omega_0}\right)^2}$$

Identify ω_0, m and K.

3) Using this expression, quickly show that the type of the filter can be found.

4) Obtain, point by point, the Bode plot of this filter.

5) Determine the expressions of the asymptotes of the Bode plot and the coordinates of their intersection.

6) Recall what a resonance is and determine in which case this filter has one.

7) Determine for which angular frequency ω_R (and frequency f_R) the gain is maximum and the corresponding phase shift.

8) Determine the –3 dB cut-off angular frequencies (and frequencies) as well as the phase shifts for each of the cut-off frequencies.

9) Calculate the width of the band pass of this filter. Then determine its resonance (or quality) factor defined as:

$$Q=\frac{\omega_R}{\omega_2-\omega_1}$$

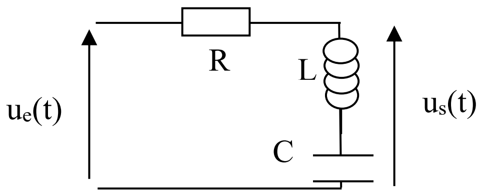
What does this factor represent for this filter?
Re-write the transfer function using Q.

10) Study qualitatively how the Bode plot of this filter changes according to the value of m (or Q).

11) Factorise the denominator of the transfer function (in the case where it is possible to do so), then quickly draw the asymptotic Bode plot and give an approximate expression of the cut-off frequencies (or angular frequencies). Discuss the validity of the approximate expressions according to the values of m (or Q).

VI Other example of a 2nd order filter

Let us consider the same circuit as in the previous exercise, but where the output voltage is taken across both the inductor L and the capacitor C:



1) By a simple qualitative analysis, determine which is the type of this filter.

2) Find the transfer function of this filter and show that it can be written in the following canonical form:

$$\underline{T}(j\omega)=K\frac{1+\left(j\frac{\omega}{\omega_0}\right)^2}{1+2jm\frac{\omega}{\omega_0}+\left(j\frac{\omega}{\omega_0}\right)^2}$$

3) What can be said by comparing this transfer function to the one of the previous exercise?

4) Find the expressions of the gain and phase shift as functions of ω.

5) Determine the –3 dB cut-off frequencies as well as the phase shift at each of these frequencies.

6) From the transfer function, deduce the differential equation that links the voltages $u_e(t)$ and $u_s(t)$.