FUNCTIONS : Differentiability and the Differential

Objectives

- **Know the definition of the derivative and its various interpretations.**
- **Calculation of derivatives.**
- **First-order approximation of standard functions.**
- **Understand the notation used in physics.**

1 Derivative

1.1 Introduction

1.1.1 Rate of change

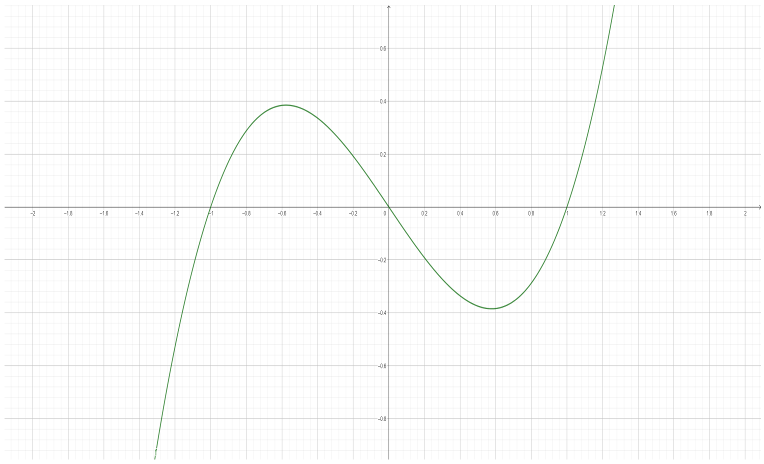
In physics, one often encounters the following quotient :

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

representing the change in a quantity f relative to the change in time t, denoted as $\dfrac{\Delta f}{\Delta t}$. This is called the rate of change of f. The letter Δ (Delta) is used to denote a variation, a large **D**ifference.

Example 1.

1. In electricity, $q(t)$ denotes the charge in coulombs on a conductor at time t, and we consider
$$\frac{q(t_1) - q(t_2)}{t_1 - t_2} = \frac{\Delta q}{\Delta t}$$
In mechanics, $x(t)$ denotes the distance traveled at time t, and we consider
$$\frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \tag{1}$$
Provide a physical interpretation of the two rates of change above.
2. In mathematics, if f is a function and x is its variable, we consider
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta f}{\Delta x}$$
Interpret the rate of change graphically using the following graph.



Example 2.

1. Write the rate of change of the functions $\sin(x)$, $\exp(x)$, and $\ln(1+x)$ at $a=0$.
2. Write the rate of change of $f(x)=\ln(x)$ at $a=1$.
3. Write the rate of change of $\sqrt{x}$ at $a=4$.

1.2 Differentiation at a point

Let a be fixed and $x \neq a$. Note that by setting $h = x - a \neq 0$ (where h represents the distance from x to a), the difference quotient at a can be rewritten solely in terms of h :

$$\frac{f(x)-f(a)}{x-a} = \frac{f(a+h)-f(a)}{h}.$$

Definition 1.
Let $a \in I$, where I is an interval of $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a function.

(i) We say that f is differentiable at a if and only if

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \text{or} \quad \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

exists and is finite; this limit is then denoted by $f'(a)$ and called the derivative of f at a.

(ii) We say that f is differentiable from the right at a if and only if

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} \quad \text{or} \quad \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

exists and is finite; this limit is then denoted by $f'_d(a)$ and called the right-hand derivative of f at a.

(iii) We say that f is differentiable from the left at a if and only if

$$\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h} \quad \text{or} \quad \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$$

exists and is finite; this limit is then denoted $f'_g(a)$ and called the left-hand derivative of f at a.

Example 3.
Let f be the absolute value function. Examine the differentiability of f at 0.

Proposition 1.
Let a be a real number belonging to an open interval I.

(i) For f to be differentiable at a, it is necessary and sufficient that f be differentiable from the right and from the left at a and that $f'_d(a) = f'_g(a)$, with both being finite numbers.

(ii) If $f'_d(a)$ and $f'_g(a)$ exist and are finite but different, then the point $A(a, f(a))$ is called a corner point (or cusp).

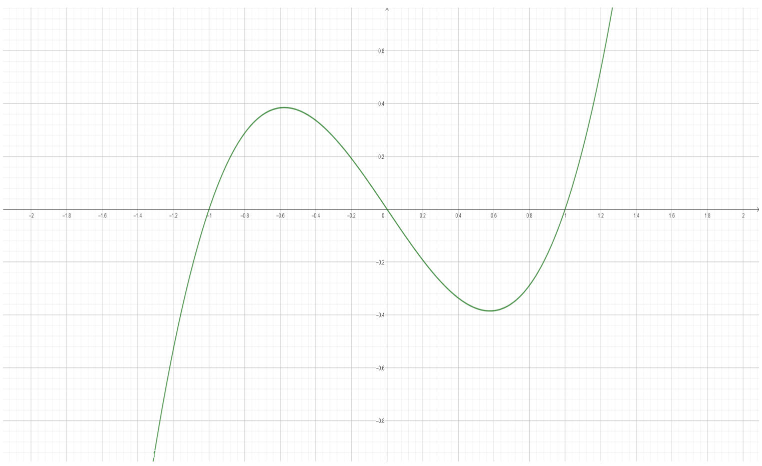
(iii) If $\displaystyle\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \infty$, then the curve has a vertical left-hand semi-tangent at a.

Remarque 1.
If I is of the form $[a; b[$, then the concept of the right-hand derivative coincides with the concept of the derivative ; there is no left-hand derivative. The same applies to an interval of the form $]b; a]$.

Example 4.
Give an example of a function for each of the preceding cases and plot them.

1.3 Graphical interpretation of the derivative

Using the previous notation, interpret the derivative of f at a graphically using the following graph.



Theorem 1 (Equation of the tangent line)**.**
Let f be a function differentiable at a. Then, an equation of the tangent line to the curve of function f at a is

$$y - f(a) = f'(a)(x - a)$$

Example 5.
Determine the equation of the tangent line to the function $f(x) = \dfrac{1}{x}$ at $x = 3$.

Proposition 2.
A function that is differentiable at a real number a is continuous at a.

Example 6.
Is the converse true ?

1.4 Application to calculating limits

The following results are easily obtained using the derivative; indeed, if one recognizes the limit of the difference quotient within the limit being sought, that limit equals the derivative for differentiable functions.

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Property 1.

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} = \cos 0 = 1$
2. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$
3. $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
4. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Example 7.
Prove these results.

Remarque 2. This technique can also be used to calculate the limit of certain quotients by expressing them as a quotient of difference quotients.

Example 8.
Find the limit as $x \to 0$ of the expression : $\dfrac{\sqrt{x+1} - 1}{\sin x}$

Remarque 3. And by combining this with a change of variable.

Example 9.
Find the limit as $x \to 0$ of the expression : $\dfrac{\sin(2x)}{x}$

1.5 Derivative function

Definition 2.
The derivative of a function $f : I \to \mathbb{R}$ is the function that assigns $f'(x)$ to each x in I. This function is denoted by f'. We say that f is differentiable on I if and only if f is differentiable for all x in I.

Example 10.
Is the square root function differentiable on its domain ?

Definition 3 (Leibniz notation)**.**
Starting from $f'(a) = \lim\limits_{x \to a} \dfrac{\Delta y}{\Delta x}$,
we **denote** it as

$$f'(a) = \frac{df}{dx}(a)$$

or alternatively $f' = \dfrac{df}{dx}$, $f'(x) = \dfrac{df}{dx}$, etc.

1.6 Operations on derivatives

Theorem 2.
Let $\lambda \in \mathbb{R}$, and let $f, g : I \to \mathbb{R}$ be two functions differentiable on I ; then :

- **(i)** $f + g$ is differentiable on I and $(f+g)' = f' + g'$
- **(ii)** λf is differentiable on I and $(\lambda f)' = \lambda f'$
- **(iii)** fg is differentiable on I and $(fg)' = f'g + fg'$
- **(iv)** If $g(x) \neq 0$ for all $x \in I$, then $\dfrac{f}{g}$ is differentiable on I and $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

Theorem 3.
Let I and J be two intervals of $\mathbb{R}$, and let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be such that $f(I) \subset J$.
Consider the function $g \circ f$ from I to $\mathbb{R}$, defined by $x \mapsto g(f(x))$.
If f is differentiable on I and g is differentiable on J, then $g \circ f$ is differentiable on I and $(g \circ f)' = (g' \circ f)f'$.

Example 11.
Let h be the function defined by $h(x) = e^{\ln x}$. Using the notation from the previous theorem, determine I, J, f, and g, and then calculate the derivative of the function h.

Remarque 4.
In physics, Leibniz notation is frequently used. Suppose, for example, we have three functions z, y, and x such that $z = y \circ x$—that is, $z(t) = y(x(t))$ where t is the variable. In Leibniz notation, the expression $z'(t) = (y \circ x)'(t) = y'(x(t)) \cdot x'(t)$ is written as :

$$\frac{dz}{dt}(t) = \frac{dy}{dX}(x(t)) \cdot \frac{dx}{dt}(t)$$

Standard practice in physics is to write the preceding equality as :

$$\frac{dz}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

This notation is useful as a mnemonic device, but it carries two potential pitfalls :

- The dx in the denominator is not the same as the dx in the numerator. The first corresponds to the variable of y, whereas the second corresponds to the function x.
- It is not explicitly stated that $\dfrac{dy}{dx}$ is evaluated at $x(t)$.

1.7 Higher-order derivatives

Definition 4.
Let $f : I \to \mathbb{R}$ be a function.
The successive derivatives of f are defined step-by-step by recurrence by setting :
For $a \in I$, $f^{(n)}(a) = \left(f^{(n-1)}\right)'(a)$ where $f^{(n)}$ is the derivative of $f^{(n-1)}$.
$f^{(n)}$ is called the n-th derivative of f.
We say that f is n times differentiable on I if and only if $f^{(n)}$ is defined on I.
We say that f is infinitely differentiable on I if and only if f is n times differentiable on I for every natural number n.

Example 12.
Determine the n-th derivative of the function f defined by $f(x) = e^{2x}$.

Remarque 5.
The following notations are also used for successive derivatives :

$$f'(x) = \frac{df}{dx}, f''(x) = \frac{d^2 f}{dx^2}, \ldots, f^{(n)}(x) = \frac{d^n f}{dx^n}$$

Theorem 4.
Let $\lambda \in \mathbb{R}$, and let $f, g : I \to \mathbb{R}$ be two functions that are n times differentiable on I. Then :

- **(i)** $f + g$ is n times differentiable on I and $(f+g)^{(n)} = f^{(n)} + g^{(n)}$
- **(ii)** λf is n times differentiable on I and $(\lambda f)^{(n)} = \lambda f^{(n)}$
- **(iii)** If $g(x) \neq 0$ for all $x \in I$, then $\dfrac{f}{g}$ is n times differentiable on I

1.8 Class of a function

Let $f : I \to \mathbb{R}$ be a function. Let $n \in \mathbb{N}$.
We say that f is of class $\mathcal{C}^n$ on I if and only if f is n times differentiable and $f^{(n)}$ is continuous on I.
We say that f is of class $\mathcal{C}^\infty$ on I if and only if f is infinitely differentiable on I.
We say that f is of class $\mathcal{C}^0$ on I if and only if f is continuous on I.

Property 2.
Rational, trigonometric, exponential, and logarithmic functions, as well as their compositions, are of class $\mathcal{C}^\infty$ on their domains of definition.

2 Differential

2.1 Differential at a point

In physics, many phenomena are modeled by considering the variation of a quantity under study when the variable undergoes a "small" change. In mathematics, this corresponds to seeking an approximation when the variable x is "close" to a fixed point a.

When x is close to a, a first, very rough approximation—assuming f is continuous at a—is to write $f(x) \approx f(a)$. To make this statement mathematically rigorous, we say that as $x \to a$:

$$f(x) = f(a) + o(1).$$

The term $o(1)$ represents the error incurred when approximating the function $f(x)$ by the constant $f(a)$. This is known as a zeroth-order approximation (involving no dependence on x) ; it is a very imprecise approximation.

Example 13. As $x \to 0$, we have $\exp(x) = 1 + o(1)$.

Let $f : I \to \mathbb{R}$ be a function differentiable at a. By definition, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a),$$

which can be rewritten, as $x \to a$, as

$$\frac{f(x) - f(a)}{x - a} = f'(a) + o(1).$$

Multiplying this equation by $(x - a)$, we obtain

$$f(x) - f(a) = f'(a)(x - a) + o(x - a),$$

or equivalently

$$f(x) = f(a) + f'(a)(x - a) + o(x - a).$$

The terms $o(x - a)$ represent error terms : this is the error incurred when approximating $f(x)$ by its tangent at x, and the preceding calculation tells us that this error is negligible compared to $(x - a)$. This is known as a first-order approximation of f around a.
We can rewrite this expression in terms of $h = x - a$:

$$f(x + h) = f(x) + f'(a)h + o(h).$$

Example 14. Write the first-order approximation of $f(x) = x^3$ at $a = 1$. Deduce an approximate value for $(1.001)^3$ and compare it to the exact value.

Definition 5.
The differential function of f at $x = a$ is defined as follows :

$$df_a : h \mapsto f'(a)h$$

Thus, the previous approximation becomes :

$$\Delta f = f(a + h) - f(a) = df_a(h) + o(h).$$

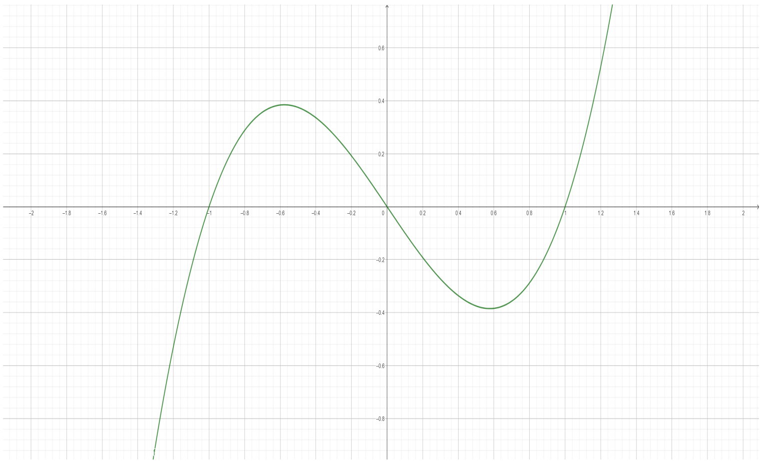
Thus, the differential df provides an approximation of the function's variation Δf as $h = \Delta x$ approaches 0.

Example 15.
Calculate the differentials of the square function and the identity function.

Remarque 6.
Note that the differential is a linear function.

2.2 Approximation of Δf by df

For a fixed a, represent the following on the graph : $h = \Delta x$, Δf, $df_a(h)$, and the error term $o(h)$.



dx **in mathematics** :
dx is the differential of the identity function $id(x) = x$

$$dx : h \mapsto h$$

thus, we have the following equality of functions :

$$df_a = f'(a)dx$$

dx in physics :
In physics, when considering a small variation of x (an infinitesimal variation of x), this variation is denoted as dx rather than Δx ; this effectively equates the function dx with the image of Δx under dx :

$$dx(\Delta x) = \Delta x$$

Note that this equality holds true regardless of the value of Δx.
It is important to note that while $\Delta x = dx$, the same does not apply to Δy and dy.

Remarque 7.

- We have seen that $\dfrac{df_a}{dx}$ can be viewed as a notation, but it can now also be interpreted as a quotient of differential functions.
- Moving from $df_a = f'(a)dx$ to $\dfrac{df_a}{dx}$ creates the impression of dividing by dx, whereas in reality, these are two different forms of expression : the first represents a relationship between two differentials, while the second is a notation for the derivative.

Example 16.
These notations require careful handling but are very useful.
Consider the area of a circle with radius R and diameter $D = 2R$
We have $S = \pi R^2$ or $S = \pi \dfrac{D^2}{4}$.
Differentiate both equations. What do you observe ?
Express dS in two different ways : first in terms of dR, then in terms of dD.

Example 17 (in electricity)**.**
A resistor R subjected to a potential difference U across its terminals has a direct current of intensity $I = \dfrac{U}{R}$ flowing through it.
What change in intensity corresponds to a small change in R? Perform the calculation for $R = 100$ ohms, a constant voltage of 100 volts, and a change $\Delta R = 1$ ohm.
This type of reasoning can also be applied to uncertainty calculations : if the value of R is known with a precision ΔR of plus or minus $1\,\Omega$, what is the resulting error ΔI in the intensity ?

2.3 Operations on differentials

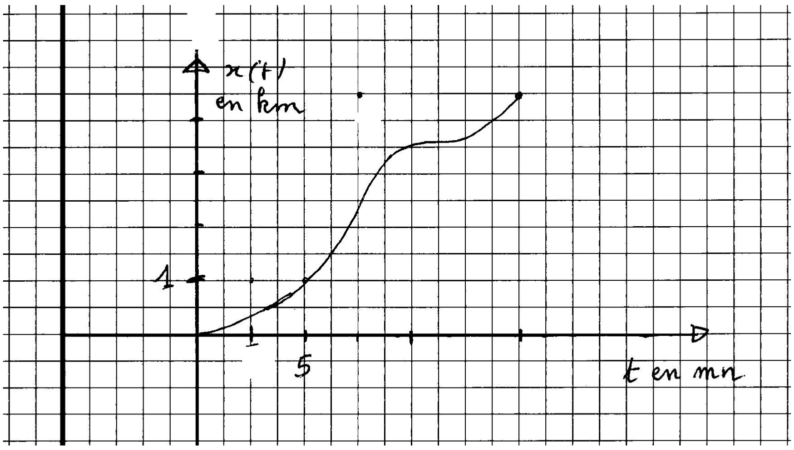
For all differential calculations, one can either return to the definition and use the rules for calculating derivatives, or directly use the following properties, which are analogous.

Theorem 5.
Let $\lambda \in \mathbb{R}$, and let $f, g : I \to \mathbb{R}$ be two functions differentiable on I ; then :

- **(i)** $f + g$ is differentiable on I and $d(f+g)_a = df_a + dg_a$
- **(ii)** λf is differentiable on I and $d\left(\lambda f\right)_a = \lambda df_a$
- **(iii)** fg is differentiable on I and $d\left(fg\right)_a = g(a)df_a + f(a)dg_a$
- **(iv)** If $g(x) \neq 0$ for all $x \in I$, then $\dfrac{f}{g}$ is differentiable on I and $d\left(\dfrac{f}{g}\right)_a = \dfrac{g(a)df_a - f(a)dg_a}{g^2(a)}$

Exercices

Exercise 1.
The curve below represents the distance x traveled by a cyclist as a function of time t (in minutes).



1. Express the cyclist's speed in terms of the variables given in the problem statement.
2. Using the graph :
   (a) What is the speed at $t = 15$ min ?
   (b) At what time(s) is the speed highest (in km/h) ?
   (c) At what time(s) is the speed lowest (in km/h) ?

Exercise 2.
Calculate $f'(x)$ for

1. $f(x) = \dfrac{\sin x}{1 - \cos x}$
2. $f(x) = \dfrac{x-1}{|x+1|}$
3. $f(x) = \dfrac{1}{\sqrt{x^3}}$
4. $f(x) = \sqrt{\dfrac{x+1}{x-1}}$
5. $f(x) = \sqrt{1 + x^2 \sin^2 x}$
6. $f(x) = \dfrac{\exp(1/x) + 1}{\exp(1/x) - 1}$.
7. $f(x) = \ln\left(\dfrac{1 + \sin(x)}{1 - \sin(x)}\right)$
8. $f(x) = (x(x-2))^{1/3}$.

Exercise 3.
Calculate the following derivatives :

1. $\dfrac{dH}{d\omega}$ with $H = \dfrac{R\omega}{1 - \omega^2}$
2. $\dfrac{di}{dR}$ with $i = \dfrac{CR^2\omega}{1 - LR}$
3. $\dfrac{dx}{dt}$ with $x = \sqrt{mt^2 + pt}$

Exercise 4.
Examine the differentiability at 0 of each of the following functions :

1. $f : x \mapsto x|x|$
2. $g : x \mapsto \dfrac{x}{1 + |x|}$
3. $h : x \mapsto \dfrac{1}{1 + |x|}$.

Exercise 5.
Let f be the function defined by : $\begin{cases} f(x) = x^2 - 1 \text{ if } x < 0 \\ f(x) = x^2 + 1 \text{ if } x \geqslant 0 \end{cases}$.
Show that at $x_0 = 0$, f is differentiable from the right but not from the left.

Exercise 6.
Extend by continuity at 0 and examine the differentiability of

1. $f(x) = \sqrt{x} \ln x$.
2. $g(x) = \dfrac{e^x - 1}{\sqrt{x}}$.

Exercise 7.

Assume for the moment that $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x^2} = -\dfrac{1}{2}$.
Consider the function f defined by : $\begin{cases} \dfrac{\cos x - 1}{\sin x} \text{ for } x \in \left]0; \dfrac{\pi}{2}\right] \\ f(0) = 0 \end{cases}$.
Study the differentiability of f on its domain and determine f'.

Exercise 8.
Given that f is a function differentiable at x_0, calculate the following limits :

1. $\lim\limits_{h \to 0} \dfrac{f(x_0 - h) - f(x_0)}{h}$
2. $\lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0 - h)}{h}$

Exercise 9.
Calculate the following limits by expressing them in terms of a rate of change (difference quotient).

1. $\lim\limits_{x \to 0} \dfrac{\ln(x+2) - \ln 2}{x}$
2. $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2+1} - 1}{x}$
3. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{\ln(x+1)}$
4. $\lim\limits_{x \to 0} \dfrac{\ln(1+x^2)}{x}$
5. $\lim\limits_{x \to 1} \dfrac{x - 1}{x^n - 1}$

Exercise 10.
Let $f(x) = \dfrac{1+x}{10+x^3}$.

1. Calculate the derivative of f.
2. Give the equation of the tangent line to f at 0.
3. Give the differential of f at 0.
4. Deduce an approximation of $f(0.1)$.

Exercise 11.
Let $f(x) = \sqrt{5 + x^2}$. Determine the differential of f at $a = 2$ and deduce an approximation of $f(2.03)$.

Exercise 12.
Calculate the differentials of the following functions :

1. $f(t) = t\ln(t)$
2. $G = \dfrac{R}{\sin R}$
3. $f = r\cos\theta$, where r and θ are functions of t.