# LINEAR DIFFERENTIAL EQUATIONS

**Learning objectives**

- **To solve a first order linear differential equation with constant coefficients**
- **To solve a second order linear differential equation with constant coefficients**

# 1 Introduction

## 1.1 In mathematics

You learned before that there exists a unique function $f$ equal to its derivative function and satisfying $f(0) = 1$. This function is the exponential function.
Thus, the equation $f' = f$ where $f$ is an unknown function, admits the exponential function $x \longmapsto e^x$ as solution. We say that $f' = f$ is a differential equation.

**Example 1.**

1. Prove that the functions $f$ defined by $f(x) = ke^x$ with $k$ a real constant are solutions of the differential equation $f' = f$.
2. Give three distinct solutions for the above equation. How many solutions do we have for this equation ?

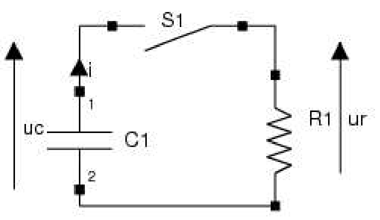
## 1.2 In physics

**Example 2.**

Let's consider this electrical network. We get the relationships :

$i = C\dfrac{dU_C}{dt}$ et $U_R = -Ri$ et $U = U_C = U_R$.

Let's deduce a differential equation checked by $U$.



# 2 Differential equations

**Definition 1.**
A differential equation is a mathematical equation that relates some function $f$ with its $n$-th derivatives. An ordinary differential equation is a differential equation containing one or more functions of one independent variable and its derivatives. The term ordinary is used in contrast with the term partial differential equation

**Example 3.**
$f'(x)f(x) + 2xf"(x) + e^x = 0$ is an ordinary differential equation.

**Notations** :

- to simplify we do not write $x$ in $f(x)$.
- in mathematics, we use the letter $y$ instead of the letter $f$. So **y is a function**.
- in mechanics, the variable is often $t$, the function is $x$ (instead of $y$) and we write $\dot{x}$ instead of $f'$ (Newton's notations).
- in physics, we use the notation $\dfrac{d^nU}{dt^n}$ fot the $n^{\text{th}}$ derivative of $U$.

**Example 4.**
Write using mathematical, mechanical and physical notations the equation of the previous example .

**To solve** a differential equation means to look for all functions satisfying the equation.
Only the simplest differential equations are solvable by explicit formulas. For instance the previous differential equation can't be solved explicitely. In this case, we use softwares. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers.

*Numerically approximation for the equation $f'(x)f(x) + 2xf"(x) + e^x = 0$ by Mathématica*

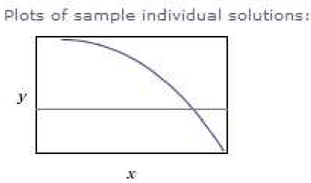



FIGURE 1 – $y(1) = 1$ et $y'(1) = 0$

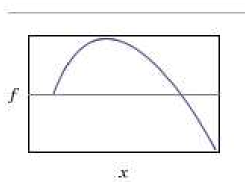



FIGURE 2 – $y(1) = 0$ et $y'(1) = 1$

**An integral curve** is a curve that represents a specific solution to an ordinary differential equation.

**Example 5.**
Sketch integral curves of the solutions in our first example.

# 3 Linear Differential equation

**Definition 2.**
A linear differential equation is an equation of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + .... + a_1(x)y' + a_0(x)y = b(x)$$

where $a_n$, ...,$a_0$ and $b$ are functions, $a_n$ is a non zero function.

- $b$ is called the right side of the equation.
- When the right side of the equation satifies $b = 0$, we say that this is **an homogeneous equation**.
- $n$ is called the order of the equation.

**Example 6.**
Among those following equations, find the linear equations. For each one, precise if it is homogeneous or not and give its order.

1. $\left(\dfrac{dx}{dt}\right)^2 + 3x = 5$
2. $e^x y^{(5)} + \ln xy = \dfrac{2x}{x+3}$
3. $x\ \dot{x} + t^2 x = 3$
4. $2y" - 3y' = y$

**Proposition 1. Structure of the set of solutions**
Solutions of a lienar differential equation are the sum of **the** solution of the homogenenus equation and of et d'**one** particular solution.

**Example 7.**
Prove the previous property for n=2.

This property is very useful to solve linear differential equations **on conditions** that we know solutions of the homogenenous equation and a solution of $(E)$ (very difficult to find in general). In this section, we will focus on a first and second order linear differential equations with constant coefficients.

**Proposition 2. Superposition principle**
Let $(H)$ be a linear homogeneous equation, let $(E_1)$, $(E_2)$ and $(E_3)$ three équations having $(H)$ as homogeneous equation and $d_1$, $d_2$ and $d_1 + d_2$ respectively as right side.
Let $f_1$ and $f_2$ be solutions respectively of $(E_1)$ and of$(E_2)$.
Then $y = f_1 + f_2$ is a solution of the equation $(E_3)$

**Example 8.**
Prove the previous property for n=2.

# 4 First order linear differential equation with constant coefficients

## 4.1 Homogeneous linear equation

**Definition 3.**
A first order linear differential equation with constant coefficients is of the form :

$$y' + ay = 0$$

where $a$ is a real constant.

**Proposition 3.**
Solutions of the equation $y' + ay = 0$ are of the form $y = ke^{-ax}$ with $k$ a real constant.

**Proof :**
We have already seen that the functions $f$ defined by $f(x) = ke^{-ax}$ are solutions of this equation. Thus it suffices to prove that there are only those functions.
Let's consider $f$ a solution of $y' + ay = 0$ and $g$ be a function such that $g(x) = f(x)e^{ax}$.
Knowing that $f$ is a solution of $y' + ay = 0$, let's dertermine $g$ and let's deduce $f$.

**Example 9.**
Solve the differential equation : $2y' - 3y = 0$.

## 4.2 Equations with a right-side

### 4.2.1 Définition

**Definition 4.**
A first order linear differential equation with constant coefficients is of the form :

$$y' + ay = b(x)$$

where $a$ is a constant and $b$ a continuous function from $I$ to $\mathbb{R}$.

### 4.2.2 Solving (E)

As we know the solution of the homogeneous equation associated, it sufficies to find a particular solution for the differential equation. We will focus on particular cases for the right-side.
**The right-side is a product of polynomial and exponential functions** Let $m \in \mathbb{R}$, $P$ be a polynomial function of degree $n \in \mathbb{N}$ and $(E) : y' + ay = P(x)e^{mx}$.
Thus a particular solution of $(E)$ is of the form $y_p = h(x)Q(x)e^{mx}$ where $Q$ is a polynomial function of degree $n$ and :

1. If $m \neq -a$, then $h(x) = 1$.
2. If $m = -a$, then $h(x) = x$.

**Example 10.**
Solve the equation :
$2y' + 3y = 2x + 1$

**Example 11. In physics**
In physics, we will have to solve : $y' + ay = K$ with $K$ a real constant and $a \neq 0$. Let's prove that the particular solution is a constant let's deduce solutions of the above equation.

**The right-side is a product of a linear combination of sine, cosine and exponential functions** Let $\alpha, \beta, \omega, m$ be four real numbers and $(E) : y' + ay = (\alpha\cos(\omega x) + \beta\sin(\omega x))e^{mx}$.
Then a particular solution of $(E)$ is of the form $y_p = (A\cos(\omega x) + B\sin(\omega x))e^{mx}$.

**Example 12.**
Solve the differential equation $y' + 4y = \cos(2x)$.

### 4.3 Equations with initials values - Cauchy's problem

**Proposition 4.**
Let $x_0 \in I$ and $y_0 \in \mathbb{R}$. There exists a unique solution $f$ of the equation $(E) : y' + ay = b$, satisfying the initial value $f(x_0) = y_0$.

**Example 13.**
Solutions of the $y' + 2y = 5$ are of the form : $y = ke^{-2x} + \dfrac{5}{2}$
Find the solution satisfying $f(1) = 2$.

### 4.4 Problems leading to a first order differential equation in physics

- In Electronics : $y' + \dfrac{1}{\tau}y = E(t)$ where $\tau > 0$ is a homogeneous constant at a time and $E(t)$ is the input signal , for example a voltage provided by a generator .
- In Chemistery : $\dfrac{dm}{dt} = -km$, weight of a reactant in a chemical reaction
- In Thermodynamics : $\dfrac{dT}{dt} = -k(T - T_0)$. K temperature of a body immersed in a medium according to Newton's law.

**Example 14.**
Solve the above equations.(We will take $E(t) = \cos t + \sin t$)

## 5 Second order linear differential equation with constant coefficients

### 5.1 Homogeneous differential equation

**Definition 5.**
A Second order homogeneous linear differential equation with constant coefficients is of the form

$$(H) : ay'' + by' + cy = 0$$

with $a$, $b$ et $c$ three real constants and $a \neq 0$.

**Proposition 5.**
Let's consider $(H)$ the equation : $(H) : ay'' + by' + cy = 0$.
Let's form **the characteristic polynomial** associated to $(H)$, and find its roots, we this get the characteristic equation $(EC) : ar^2 + br + c = 0$.
We denote $\Delta = b^2 - 4ac$ its discriminant.
The general solution is described by three cases :

1. If $\Delta > 0$, $(EC)$ has two distinct real roots $\alpha$ and $\beta$, thus general solutions of $(H)$ are $f(x) = Ae^{\alpha x} + Be^{\beta x}$ with $A$ and $B$ two real constants.
2. If $\Delta = 0$, $(EC)$ has one real root$\alpha$, so the general solutions of $(H)$ are $f(x) = (Ax + B)e^{\alpha x}$ where $A$ and $B$ are two real constants.
3. If $\Delta < 0$, $(EC)$ has two complex conjugate roots $\alpha + i\beta$, and $\alpha - i\beta$ so the general solutions of $(H)$ are $f(x) = e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right)$ where $A$ and $B$ are two real constants.

**Example 15.**
Find a particular solution for those equations :

1. $y'' - 5y' + 10y = 0$
2. $y'' - 4y' + 4y = 0$
3. $y'' - y' - 2y = 0$

## 5.2 Equations with right-side

### 5.2.1 Definition

**Definition 6.**
A second order linear differential equation $(E)$ with constant coefficients is of the form :

$$ay'' + by' + cy = d(x)$$

où $a$, $b$ and $c$ are three real constants, $a \neq 0$ and $d$ is a continuous function from an interval $I$ to $\mathbb{R}$.

### 5.2.2 Solving (E)

As we know the solution of the homogeneous equation associated, it sufficies to find a particular solution for the differential equation. We will focus on particular cases for the right-side.
**The right-side is a product of polynomial and exponential functions**
Let $m \in \mathbb{R}$, $P$ be a polynomial function of degree $n \in \mathbb{N}$ and $(E) : ay'' + by' + cy = P(x)e^{mx}$ with $a \neq 0$.
Then a particular solution of $(E)$ is of the form $y_p = h(x)Q(x)e^{mx}$ with $Q$ a polynomial function of degree $n$ and :

1. If $m$ is not a solution of $(EC)$, then $h(x) = 1$.
2. If $m$ is a root of $(EC)$ and $\Delta \neq 0$ ( $m$ is a single root of $(EC)$), then $h(x) = x$.
3. If $m$ is a root of $(EC)$ and $\Delta = 0$ ($m$ is a double or repeated root of $(EC)$), then $h(x) = x^2$.

**Example 16.**
Find a particular solution for the equations :

1. $y'' + y' - 2y = (x + 1)$

**The right-side is a product of a linear combination fo sine, cosine and exponential functions** Let $\alpha, \beta, \omega, m$ be four reals and $(E) : ay'' + by' + cy = (\alpha\cos(\omega x) + \beta\sin(\omega x))e^{mx}$.
Then a particular solution of $(E)$ is of the shpae $y_p = h(x)(A\cos(\omega x) + B\sin(\omega x))e^{mx}$ and :

1. If $m + i\omega$ is not a root of $(EC)$ then $h(x) = 1$.
2. If $m + i\omega$ is a root of $E$, then $h(x) = x$.

**Example 17.**
Solve the following equations :

1. $y'' + y = \cos x$
2. $y'' + y' - 2y = \sin(2x)$

## 5.3 Cauchy's Problem, initail values problem

**Proposition 6.**
Let $x_0, y_0, x_1, y_1 \in \mathbb{R}$. There exists a unique solution $f$ of the equation
$(E) : ay'' + by' + cy = d(x)$, satisfying initial values $f(x_0) = y_0$ and $f'(x_1) = y_1$.

**Example 18.**
Solutions of the equation $y'' + 3y' + 2y = x + 1$ are of the form : $y = Ae^{-x} + Be^{-2x} + \frac{1}{2}x - \frac{1}{4}$
Find the solution satisfying $f(0) = 1$ and $f'(0) = 2$

**Remark 1.**
La solution $f$ of a second order homogeneous differential equation has two constants, thus it sufficies to have two initial values to compute those constants. Those conditions may concern $f$ and $f'$, or we may have two conditions on $f$.

**Example 19.**
Let's consider an infinite length of bar which is embedded in a wall. It is considered that the wall has a temperature $\theta_M$ above ambient air temperature $\theta_A$. It is assumed that the stabilized temperature $\theta$ to $x$ distance of the wall satisfies the differential equation $\frac{d^2\theta}{dx^2} - m^2\theta = -m^2\theta_A$, with $m$ a positive constant.
Let's express $\theta$ with the previous data.

## 5.4 Problems leading to a differential equation of second order in physics

- In Electronics : RLC network : $LC\frac{d^2U}{dt^2} + RC\frac{dU}{dt} + U = U_O \sin(\omega t)$
- In Mechanics : Restoring force of a spring : $-mg - ky - K\frac{dy}{dt} = m\frac{d^2y}{dt^2}$

# Exercises

**Exercise 1.**
Among those equations, find the linear equations ?

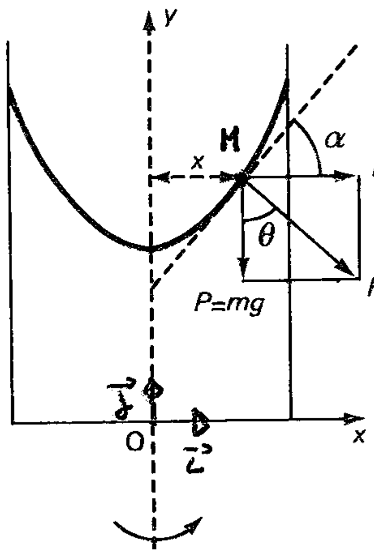
1. $2yy' + 3 = t$
2. $x^2y' + e^x yy' = \sin 5x$
3. $t\dot{x} + 3t^2 = \ln t$
4. $x\dfrac{dx}{dt} + 5x = 3t$

**Exercise 2.**
Solve the following equations :

1. $-2y' + 5y = x^2 - x + 3$
2. $3\dfrac{dz}{dt} = 5z + \sin t$
3. $-2x + \dot{x} = \cos(3t) + 2\sin(3t)$
4. $t + e^t + q = \dfrac{dq}{dt}$

**Exercise 3.**
Sketch the integral curves $C_n$ of the functions $f_n$ solutions of $y' - y = 0$ and satisfying $f_n(0) = n$ for all $n \in \mathbb{Z}$.

**Exercise 4.**




Let's consider a vertical cylindrical tube containing a liquid and rotating quickly about its axis with a constant angular speed $\omega$. Our goal is to determine the curve (C).
Let $(\mathcal{C})$ be the curve, which is the intersection of a surface and a plane containing the symmetry axis of the tube. The origin is on the symmetry axis of the tube (see picture)
Let $M(x;y)$ be a point on the surface of mass $m$. There exists two forces on M

- its weight $P = mg$, which is vertical ;
- the centrifugal force m$\omega^2$x, which is horizontal.

The rotational speed is constant, the liquid surface is stable, and therefore the resultant F of the two previous forces, is perpendicular to the plane tangent to the curve in M.

1. Justify that $\alpha = \theta$

2. Justify that : $\tan(\alpha) = \dfrac{dy}{dx}$ et $\tan(\theta) = \dfrac{m\omega^2 x}{mg}$.
3. Determine the differential $(E)$ checked by $y$.
4. Solve (E).
5. Numerical computation : $\omega = 10rad/s$, $g = 10m/s^2$
   Ray tube : $10^{-2}$ m
   the liquid height, before rotation : $10^{-1}$m
   Knowing that
   the volume of the solid begotten by the rotation around the y-axis of $(\mathcal{C})$ delimited by (C),the x-axis, the y-axis and by the straight line of equation x=$10^{-2}$, is equal to $\dfrac{5\pi}{2}10^{-8}$
   and that the volume keeps constant,
   let's determine the solution of (E).

**Exercise 5.**
Solve those differential equations :

1. $9y'' + 12y' + 4y = 0$
2. $y'' + y' - 2y = 0$
3. $y'' + y' + 2y = 0$

**Exercise 6.**

1. Find a second order linear diffrential equation with constant coefficents having $e^{2x}$ and $e^x$ as solutions.
2. Find a second order linear diffrential equation with constant coefficents having $e^x$ and $xe^x$ as solutions.
3. Find a second order linear diffrential equation with constant coefficents having 1 and $x$ as solutions.
4. Find a second order linear diffrential equation with constant coefficents having $\cos 3x$ and $\sin 3x$ as solutions.
5. Find a second order linear diffrential equation with constant coefficents having $e^{2x}\cos x$ and $e^{2x}\sin x$ as solutions.

**Exercise 7.**
Finf the general solution of the following second order linear differential equations

1. (a) $y'' + 2y' + 5y = 5x^2 + x + 1$
   (b) Find the solution $f$ satisfying $f(0) = 2$ and $f'(0) = 1$.
2. $y'' + 4y = \cos 2x$
3. $y'' - 3y' + 2y = 3x + \sin x$

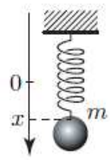
# Exercice 8 : the free oscillator

## 1 Introduction

We thus call the physical devices leading to a homogeneous linear differential equation of order 2. Here are two examples :

1. Consider a mass $m$ suspended from a spring of stiffness $k$. If the mass is displaced from its equilibrium position (in the vertical direction), its motion is governed by the differential equation

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (1)$$




   , where $c$ is a damping coefficient (due for example to friction).
2. When a capacitor of capacitance $C$ in an inductor $L$ and resistance $R$ is discharged, its charge $q(t)$ satisfies the differential equation $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = 0$.

In the following we will study the behavior of the solutions by taking the example of the pendulum.

## 2 Temporal study of the homogenuous equation

### 2.1 undamped oscillations : $c = 0$

1. Solve (1) with $c = 0$.
2. Writing the $y$ solution with a single term, describe the temporal response. Is this case observable in practice ?

### 2.2 damped Oscillations : $c > 0$

Depending on the sign of the discriminant of the characteristic equation we can distinguish 3 cases

#### 2.2.1 Low damping : $0 < c^2 < 4mk$

1. Solve (1) in that case
2. Is the mouvement perdiodic ?
3. Is there a pendulum oscillation ? If so, what can we say about their amplitudes ?
4. Give the sketch of some solutions with $x(0) = x_0$ by varying $\dot{x}(0) = v_0$

#### 2.2.2 High damping : $c^2 > 4mk$

1. Solve (1) in that case
2. What is known of the sign of the solutions of the characteristic equation ? (indication : we will look at their product and their sum)

3. Is there a pendulum oscillation ?
4. can the pendulum pass through its position of balance ?
5. Study the curve in the following two cases : $x'(0) = 0$ and $x'(0) = -4$ considering both :$m = 1, c = 5, k = 6$ and $x(0) = 1$.

### 2.2.3 critical damping : $c^2 = 4mk$

1. Solve (1) in that case
2. can the pendulum pass through its position of balance ?
3. Give the sketch of some solutions with $x(0) = x_0$ by varying $\dot{x}(0) = v_0$

In some electronic circuits, the values of the components are calculated so that the device exhibits this type of operation.

# 3 Forced Oscillation

Consider a mechanical oscillator on which a variable force $f(t)$, or a resistance-capacitance-inductance circuit fed by a variable voltage generator $E(t)$ is actuated. The response $x(t)$ to *the excitation* is the solution of a differential equation

$$m\ddot{x} + c\dot{x} + kx = b(t) \tag{2}$$

We put in the frequent case of a second member of the form $E\cos\alpha t$ (avec $E$ a positive constant)
**Unamortized case :** Solve the complete equation in the non-amortized case by distinguishing 2 cases. We will take $\omega = \sqrt{\dfrac{k}{m}}$

1. $\alpha \neq \pm\omega$

(a) Solve the complete equation with the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$.

(b) Using the formula $\cos p - \cos q = -2\sin\left(\dfrac{p+q}{2}\right)\sin\left(\dfrac{p-q}{2}\right)$ Write that solution has a product. If we consider the case $\alpha$ close to $\omega$ a **beat phenomenon** happens. Indeed we have then $|\alpha - \omega|$ small and $\alpha + \omega$ big. Considering the periods of the 2 sine of the product give the appearance of the curve solution.

2. $\alpha = \pm\omega$

(a) Solve the complete equation with the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$.

(b) Describe and give the look of the curve solution. In this case it is said that the excitation causes the resonance of the oscillator.

The fields where resonance occurs are innumerable : child swing, but also acoustic resonances of musical instruments, resonance of tides, orbital resonance in astronomy, resonance of the basilar membrane in the hearing phenomenon, resonances in circuits electronics and finally : all systems, assemblies, mechanical parts are subjected to the phenomenon of resonance. Abstract systems are also subject to resonances : one can, for example, mention the dynamics of populations. In the field of civil engineering, this phenomenon can be observed mainly in pedestrian footbridges subjected to military marches, for example, or, more generally, in structures subjected to an earthquake.

In 1850, a troop crossing in close order the bridge of the Basse-Chaine, bridge suspended on the Maine in Angers, provoked the rupture of the bridge by resonance and the death of 226 soldiers. However, the military regulations already prohibited walking on a bridge, which suggests that this phenomenon was known before.