FUNCTIONS : Derivative and Differential

Learning Ojectives

- **To become familiar with derivative number and its interpretations.**
- **To use differentials**
- **To understand notations used in physics.**

1 Derivative

1.1 Introduction

1.1.1 Difference Quotient

Consider a function f of one variable t. Suppose t changes from an initial value t_1 to a final value t_2. Then the increment of t is defined to be the amount of change in t. It is denoted by $\Delta t = t_2 - t_1$. That is $t_2 = t_1 + \Delta t$. As t changes from t_1 to t_2, f changes from $f(t_1)$ to $f(t_2)$. Thus we have the increment of f :$\Delta f = f(t_2) - f(t_1)$. In physics, we usually get this expression :

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

variation of a magnitude f to the time variation t. This is denoted by $\dfrac{\Delta f}{\Delta t}$.
This ratio is called the difference quotient of f. We use the letter Δ (Delta) to express a big **Di**fférence.

Example 1.

1. In electricity, let $q(t)$ be the electric charge in coulomb at time t and let's consider :

$$\frac{q(t_2) - q(t_1)}{t_2 - t_1} = \frac{\Delta q}{\Delta t}$$

   In mechanics, let $x(t)$ be the covered distance at time t and let's consider

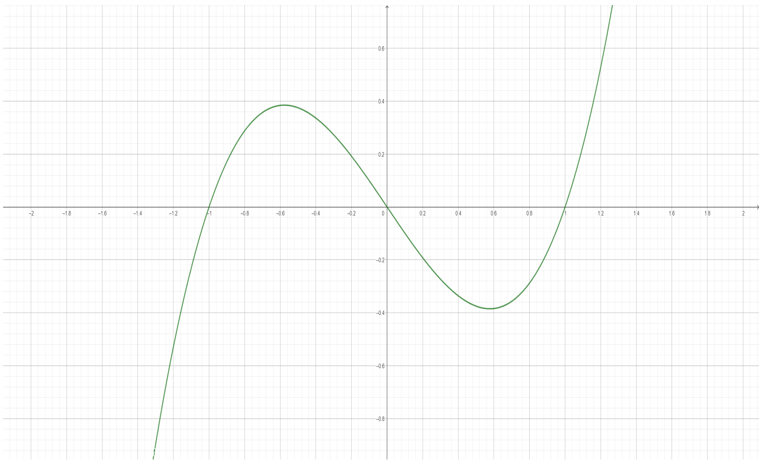
$$\frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \tag{1}$$

   What does it represent in physics ?

2. In mathematics, let f be a function and x its variable, let's consider :

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta f}{\Delta x}$$

   Give a graphic interpretation of the difference quotient using this graph.



1.1.2 The limit of the difference quotient

We have the difference quotient (**??**), what does it represent if the variation Δt is infinitely small ?
Theorically, this leads to consider the limit of the difference quotient when t_1 approaches t_2.
Thus we get :

$$\lim_{t_1 \to t_2} \frac{f(t_1) - f(t_2)}{t_1 - t_2} \tag{2}$$

Example 2.

1. In physics, how would you compute experimentally this limit ?
2. In mathematics, what can you say ?
3. Give the name of those three magnitudes when t_1 approaches t_2 (in mathematics), or when x_1 approaches x_2 (in mathematics).

1.2 Derivative at a point

Let's consider this definition (**??**) with a slight change in notation $t_2 = a$ and $h = t_1 - t_2$. Thus, the previous limit can be written using only a and h. Thus we have two ways to compute the derivative of f at a :

Definition 1.
Let $a \in I$ with I an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ a function.

(i) f is said to be differentiable at a if and only if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite ; this limit is denoted $f'(a)$ and called the derivative of the function f at a ot the differential coefficient of f at the point a.

(ii) f is right differentiable at a if and only if

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. This limit is denoted by $f'_d(a)$ and is called the right derivative of f at a.

(iii) f is left differentiable at a if and only if

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$$

exitst and is finite. This limit is denoted by $f'_g(a)$ and called the left derivative of f at the point a.

Example 3.
Let f be the absolute value function. What about the derivative of f at 0 ?

Proposition 1.
Let a be a real number belonging to an open interval I.

(i) f is differentiable at the point a if and only if both the right derivative $f'_d(a)$ and the left derivative $f'_d(a)$ both exist, are equal $f'_d(a) = f'_g(a)$, and are finite real numbers.

(ii) If $f'_d(a)$ and $f'_g(a)$ both exist and are finite then the point $A(a, f(a))$ is called a corner.

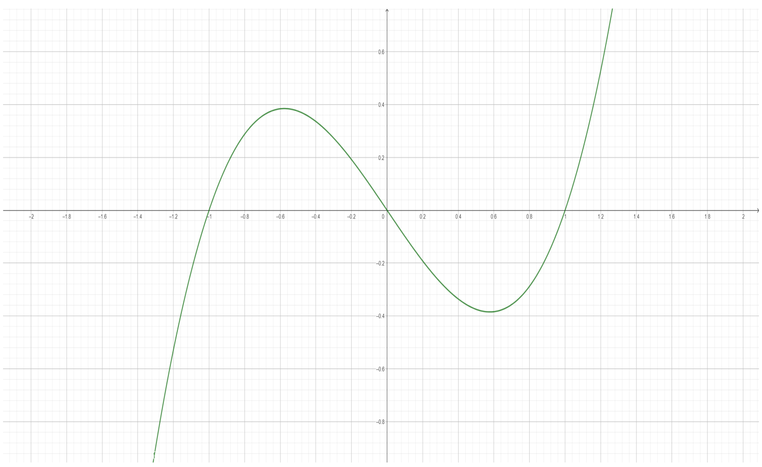
(iii) If $f'_d(a)$ and $f'_g(a)$ both exist but are infinite, then the point $A(a, f(a))$ is called a cusp.

Remark 1.
If I is of the form $[a; b[$ then the right derivative coincides with the derivative, moreover there is no left derivative. Idem for $]b; a]$.

Example 4. For each previous case, give an example of a function and draw it.

1.3 Geometrical Meaning of the derivative

With the same notations as before and using the graph below, give the geometrical meaning of the differential coefficient.



Theorem 1 (Equation of the tangent line)**.**
Let's assume that f is a differentiable function at a, then the equation of the tangent line to the curve of f at the point a is

$$y - f(a) = f'(a)(x - a)$$

Example 5.
Give the equation of the tangent line of f with $f(x) = \dfrac{1}{x}$ at the point $x = 3$

Proposition 2.
Every differentiable function at a real number a is continuous at a.

Example 6.
Is the converse true ?

1.4 Application to the calculation of limits

The following results are easily obtained using the derivative number, in fact if we recognize the limit of the difference quotient in the limit we are looking for, it is equal to the derivative number for the differentiable functions.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Proposition 3.

1. $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin x - \sin 0}{x - 0} = \cos 0 = 1$
2. $\displaystyle\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
3. $\displaystyle\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$
4. $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Example 7.
Demonstrate these results.

Remark 2. We can also use this technique to calculate the limit of certain quotients by revealing a quotient of the rate of increase.

Example 8.
Find the limit at 0 of the expression : $\dfrac{\sqrt{x+1} - 1}{\sin x}$

Remark 3. And combining with a change of variable.

Example 9.
Find the limit at 0 of the expression : $\dfrac{\sin(2x)}{x}$

1.5 The derivative function

Definition 2.
Let's consider $f : I \to \mathbb{R}$, its derivative function (also called derivative) is the function which associates to each member x of I, the derivative (or differential coefficient) $f'(x)$. This function is denoted by f'. f is said to be differentiable on I if and only if f is differentiable at all points x of I. Differentiation is the action of computing a derivative.

Example 10.
Is the square root function differentiable on its domain of definition ?

Definition 3 (Leibniz's notations)**.**
From $f'(a) = \lim_{x\to a} \frac{\Delta y}{\Delta x}$
we denote **note**

$$f'(a) = \frac{df}{dx}(a)$$

or $f' = \frac{df}{dx}$, $f'(x) = \frac{df}{dx}$,etc...

1.6 Operations on derivatives

Theorem 2.
Let's consider $\lambda \in \mathbb{R}$, $f, g : I \to \mathbb{R}$ two differentiable functions on I, yhen :

- **(i)** $f + g$ is differentiable on I and $(f + g)' = f' + g'$
- **(ii)** λf is differentiable on I and $(\lambda f)' = \lambda f'$
- **(iii)** fg is differentiable on I and $(fg)' = f'g + fg'$
- **(iv)** If for all x in I, $g(x)$ is distinct from 0 then $\frac{f}{g}$ is differentiable on I and $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Theorem 3.
Derivative of a composite function : Let I, J be two real intervals and $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ such that $f(I) \subset J$.
Let's define the function $g \circ f$, from I to $\mathbb{R}$ by $x \mapsto g(f(x))$.
If f is differntiable on I and if g is differntiable on J then gof is differentiable on I and $(gof)' = (g' \circ f)\, f'$.

Example 11.
Let's define the function h by $h(x) = e^{\ln x}$. With the previous notations, let's determine I, J, f and g to compute the derivative function of h.

Remark 4.
In physics, we use Leibniz notations. Let's assume that we have three functions z, y and x such that $z = y \circ x$, which means $z(t) = y(x(t))$ with t the variable. Using Leibniz's notations, $z'(t) = (yox)'(t) = y'(x(t)).x'(t)$ is written :

$$\frac{dz}{dt}(t) = \frac{dy}{dX}(x(t)).\frac{dx}{dt}(t)$$

This equality commonly becomes

$$\frac{dz}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$$

dx at the denominator is not the same as dx at the numerator. The first one matches the variable of y, while the second one matches the function x. Thus this writting is intuitive but dangerous.

1.7 Higher orders derivatives

Definition 4.
Let $f : I \to \mathbb{R}$ be a function.
We define higher orders derivatives of f by induction :
For $a \in I$, $f^{(n)}(a) = \left(f^{(n-1)}\right)'(a)$ where $f^{(n)}$ is the derivative function of$f^{(n-1)}$
$f^{(n)}$ is called the n^{th} order derivative of f.
f is said n times differentiable on I if and only if $f^{(n)}$ is defined on I.
f is said infinitiley differntiable on I if and only if f is n times differentiable on I for all natural number n.

Example 12.
Let's compute the n^{th} order derivative of f defined by $f(x) = e^{2x}$.

Remark 5.
We also find those following notations for higher orders derivatives :

$$f'(x) = \frac{df}{dx}, f"(x) = \frac{d^2 f}{dx^2} ... f^{(n)}(x) = \frac{d^n f}{dx^n}$$

Theorem 4.
Let's consider $\lambda \in \mathbb{R}$, $f, g : I \to \mathbb{R}$ two real valued functions n times differentiable on I, then :

(i) $f + g$ is n times differentiable on I and $(f + g)^{(n)} = f^{(n)} + g^{(n)}$

(ii) λf is n times differentiable on I and $(\lambda f)^{(n)} = \lambda f^{(n)}$ (Leibniz's formula).

(iii) If for all x in I $g(x)$ is distinct from 0, then $\dfrac{f}{g}$ is n times differentiable on I.

1.8 Differentiability class of a function

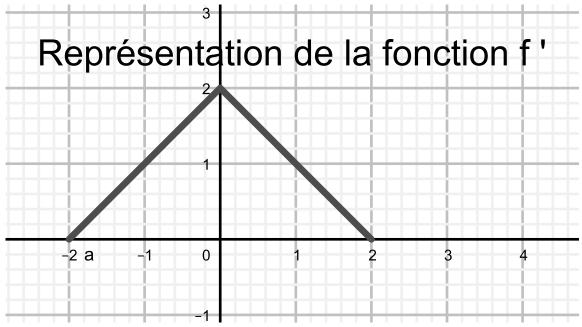
Let $f : I \to \mathbb{R}$ be a function and let $n \in \mathbb{N}$.
f is said to be of differentiability class $\mathcal{C}^n$ on I if and only if f is n times differentiable and its n-th order derivative $f^{(n)}$ is continuous on I.
f is said to be of infinitely differentiability class $\mathcal{C}^\infty$ on I if and only if f is infinitely differentiable on I. We say that f is smooth.

Example 13. Prove that the square root function is of differentiability class $\mathcal{C}^0$ on its domain of definition but is not of differentiability class $\mathcal{C}^1$.

Example 14. Let f be a continuous function on [-2,2], which representation of f' is the following.

1. Is f of class C^1 ?

2. Is f of class C^2 ?



Property 1. Rational functions, trigonometric, exponential , logarithmic functions and their composite are infinitely differentiable $\mathcal{C}^\infty$ on their domain of definition.

1.9 Limit of the derivative function at a point

To study the differentiabilityof a function at a point, we are used to computing the difference quotient. There exists another method with this theorem :

Theorem 5. Let I be a real interval, a an element of I, f a function of differentiability class $\mathcal{C}^1$ on the set $I \setminus \{a\}$ which is continuous at the point a. If its derivative function f' admits a limit l at a, then $\dfrac{f(x)-f(a)}{x-a}$ has the same limit l when the variable approaches a, thus :

1. If $\lim_{x\to a} f'(x) = l \in \mathbb{R}$ then f is differentiable at a and $f'(a) = l$ so $f \in \mathcal{C}^1(I)$
2. If $\lim_{x\to a} f'(x) = \pm\infty$ then f is not differentiable at a and its curve has a vertival tangent at the point $(a, f(a))$
3. Si $\lim_{x\to a^+} f'(x) = l_1$ et $\lim_{x\to a^-} f'(x) = l_2$ avec $l_1 \neq l_2$ then f is not differentiable at a, and the graph of f has two half-tangent of respective slopes l_1 and l_2.
4. If $\lim_{x\to a} f'(x)$ does not exist, we can't say anything about the differentiability of f at a.

Remark 6. In the first two cases, this theorem allows to draw conclusions, but not in the third case for which we have to study the difference quotient $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$.

Example 15. Let's define g by $\mathbb{R}$ par : $g(x) = \begin{cases} x^2 \sin\dfrac{1}{x} \text{ si } x \neq 0 \\ \text{g}(0) = 0 \end{cases}$.

Prove that g' has no limit at 0, while g is differentiable at 0.

2 Differential of a function

2.1 Differential at a point

Let $a \in I$ with I an interval of $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a function, differentiable at a then we get

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

thus

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a) = 0$$

which may also be written as

$$\frac{f(x) - f(a)}{x - a} - f'(a) = \varepsilon(x) \text{ avec } \lim_{x \to a} \varepsilon(x) = 0$$

and $\Delta x = x - a$ et $\Delta y = \Delta f = f(x) - f(a)$, finally we have

$$\Delta y = \Delta x f'(a) + \Delta x.\varepsilon(x) \text{ avec } \lim_{\Delta x \to 0} \varepsilon(x) = 0 \tag{3}$$

Definition 5.
The differential of a function f at a point a is defined as follows :

$$df_a : x \to f'(a)x$$

Ainsi (**??**) devient :

$$\Delta y = df_a(\Delta x) + \Delta x.\varepsilon(x)$$

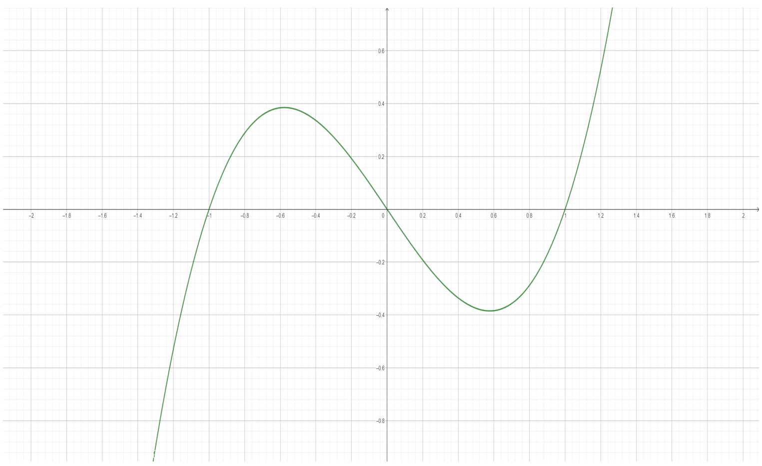
We may also write $df_a = dy$

Example 16.
Finf the differential of the square function and of the identity function.

Remark 7.
We note that the diffrential of a function is a linear map.

2.2 Approximation of Δy by dy

Draw on the picture below : Δx, Δy, $dy(\Delta x)$ et $\Delta x \varepsilon(x)$.



dx in mathematics :
dx is the differential function of the identity function$id(x) = x$

$$dx : x \to x$$

thus we have this equality of functions

$$df_a = f'(a)dx$$

dx in physics :
In physics, when the increment of x is very small (infinitesimal) we denote this increment by dx instead of Δx. This which means that we mix up the function dx and the image of Δx under this function dx :

$$dx(\Delta x) = \Delta x$$

This equality is true whatever is Δx.
It is important to note that $\Delta x = dx$, however we don't have the same equality for Δy and dy.

Theorem 6.
$\Delta y \simeq dy$ when Δx approaches 0.

Proof 1.
Let's write the relationship between Δy and dy, let's deduce the approximation.

Thus the differential function dy is an **approximation** of the increment of the function Δy when Δx approaches 0.

Remark 8.

- So $\dfrac{df_a}{dx}$ can be viewed as a notation or as a qotient of two diffentials of functions.
- From $df_a = f'(a)dx$ to $\dfrac{df_a}{dx}$, we get the impression that we divide by dx, but those are just two different notations : one is a notation between two differential of functions, while the other notation refers to the derivative.

Example 17.
Those notations are useful but we have to be careful.
Let's consider the surface of a circle of radius R and of diameter $D = 2R$.
Thus we get $S = \pi R^2$ or $S = \pi \dfrac{D^2}{4}$.
Differentiate those equalities. What do you think ?
Express dS in two different ways, in terms of dR and in terms of dD.

Example 18 (in electricity)**.**
A resistance R, with at its terminals a potential difference U, is traversed by a current DC with an intensity

$$I = \frac{U}{R}$$

.

- What is the increment of the intensity for a infinitesimal increment of U ? Compute it with $R = 100\Omega$ and an increment of U equal to 1 volts.
- What is the increment of the intensity for a infinitesimal increment of R ? Compute it with $R = 100\Omega$, $U = 100V$ and an increment $\Delta R = 1$ ohms .

2.3 Operations on differentials of functions

To compute differential of functions, we may use definitions and rules to compute derivatives or we may use the following properties (which are similar) :

Theorem 7.
Let $\lambda \in \mathbb{R}$, $f, g : I \to \mathbb{R}$ two differentiable functions on I, then :

(i) $f + g$ is differentiable on I and $d(f + g) = df + dg$

(ii) λf is differentiable on I and $d(\lambda f) = \lambda df$

(iii) fg is differentiable on I and $d(fg) = df.g + f.dg$

(iv) If for all x in I $g(x)$ is distinct from 0 then $\dfrac{f}{g}$ is differentiable on I and $d\left(\dfrac{f}{g}\right) = \dfrac{df.g - f.dg}{g^2}$

3 Logarithmic Differential

Definition 6. Let f be a differentiable. For all x such that $f(x) \neq 0$ the function $\ln|f(x)|$ is differentiable.
The logarithmic differential of f at a is the differential function of $\ln|f|$ at a.

Theorem 8.

$$d\ln|f|_a = \frac{df_a}{f(a)}$$

Example 19.
Compute the logarithmic differential of each following function at x :
$f(x) = e^{\sin(x)}$ et $g(x) = 2x + 1$.

The logarithmic differential $\dfrac{dy}{y}$ is an approximation of the **relative increment** $\dfrac{\Delta y}{y}$ whenthe increment of x is Δx. It is used for **uncertainty calculus** in physics.

3.1 Product and quotient of logarithmic differentials

Theorem 9. Let f and g be differentiables functions such that $f(x) \neq 0$ and $g(x) \neq 0$

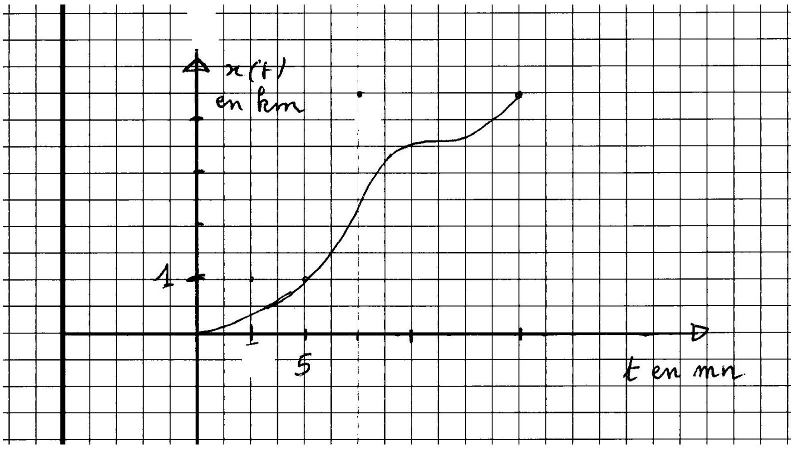
(i) If $y = f.g$ then $\dfrac{dy}{y} = \dfrac{df}{f} + \dfrac{dg}{g}$

(ii) If $y = \dfrac{f}{g}$ then $\dfrac{dy}{y} = \dfrac{df}{f} - \dfrac{dg}{g}$

Proof 2. Prove the relationship for the product :

Exercises

Exercise 1.
The curve below represents the distance x traveled by a cyclist versus time t in minutes .



1. Give the expression of the velocity of the cyclist (using notations given in the text).
2. Graphically :

(a) Let's determine its velocity at $t = 15'$?
(b) When do we have the highest velocity (in km/h) ?
(c) When do we have the lowest velocity (in km/h) ?

Exercise 2.
Compute $f'(x)$ for

1. $f(x) = \dfrac{\sin x}{1 - \cos x}$
2. $f(x) = \dfrac{x-1}{|x+1|}$
3. $f(x) = \dfrac{1}{\sqrt{x^3}}$
4. $f(x) = \sqrt{\dfrac{x+1}{x-1}}$
5. $f(x) = \sqrt{1 + x^2 \sin^2 x}$
6. $f(x) = \dfrac{\exp(1/x) + 1}{\exp(1/x) - 1}$.
7. $f(x) = \ln\left(\dfrac{1 + \sin(x)}{1 - \sin(x)}\right)$
8. $f(x) = (x(x-2))^{1/3}$.

Exercise 3.
Are the following functions differentiable at 0 :

1. $f : x \mapsto x|x|$
2. $g : x \mapsto \dfrac{x}{1 + |x|}$
3. $h : x \mapsto \dfrac{1}{1 + |x|}$.

Exercise 4.
Let's define f by $: \begin{cases} f(x) = x^2 - 1 \text{ si } x < 0 \\ f(x) = x^2 + 1 \text{ si } x \geqslant 0 \end{cases}$.
Prove that at $x_0 = 0$, f is right differentiable but not left differentiable.

Exercise 5.
Extend by continuity at 0 and study the differentiabillity

1. $f(x) = \sqrt{x} \ln x$.
2. $g(x) = \dfrac{e^x - 1}{\sqrt{x}}$.

Exercise 6.
Let's assume that f is differentiable at x_0, compute the following limits :

1. $\lim_{h \to 0} \dfrac{f(x_0 - h) - f(x_0)}{h}$
2. $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0 - h)}{h}$

Exercise 7.
Calculate the following limits using a difference quotient.

1. $\lim_{x \to 0} \dfrac{\ln(x+2) - \ln 2}{x}$
2. $\lim_{x \to 0} \dfrac{\sqrt{x^2+1} - 1}{x}$
3. $\lim_{x \to 0} \dfrac{e^x - 1}{\ln(x+1)}$
4. $\lim_{x \to 0} \dfrac{\ln(1+x^2)}{x}$

Exercise 8.
Compute the derivatives

1. $\dfrac{dH}{d\omega}$ avec $H = \dfrac{R\omega}{1 - \omega^2}$
2. $\dfrac{di}{dR}$ avec $i = \dfrac{CR^2\omega}{1 - LR}$
3. $\dfrac{dx}{dt}$ avec $x = \sqrt{mt^2 + pt}$

Exercise 9.
We admit for now that $\lim_{x \to 0} \dfrac{\cos x - 1}{x^2} = \dfrac{1}{2}$

Let's define the function f by : $\begin{cases} \dfrac{\cos x - 1}{\sin x} \text{ pour } x \in \left]0; \dfrac{\pi}{2}\right] \\ f(0) = 0 \end{cases}$.

Let's study the differentiability of f on its domain of definition and get f'.

Exercise 10.
Is the function $f : x \mapsto \cos\left(\sqrt{x}\right)$ differentiable at 0 ? C^1 at 0 ?

Exercise 11. Let's define $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto e^x \text{ si } x < 0 \\ x \mapsto ax^2 + bx + c \text{ sinon} \end{cases}$

Find a, b, c such that f is C^2 (and C^3 ?).

Exercise 12.
Calculate the following differentials :

1. $f(t) = t\ln(t)$
2. $f(t) = \dfrac{t}{\sin t}$
3. $f = r\cos\theta$, r and θ are functions depends on t.

Exercise 13. The inductance of a coil is in henrys :

$$L = \frac{4\pi N^2 S}{l.10^9}$$

S section of the coil in cm^2, l, its length in cm, et N its number of turns .
Compute an approximation for ΔL, if l increases of 1 cm knowing that :
$N = 500$, $S = 500$, $l = 50$.

Exercise 14. In a coil, the offset of the current on the voltage , AC , is given by $\tan\varphi = \dfrac{L\omega}{R}$ et $\omega = 2\pi f$.
We have $L = 100$ H, $R = 50$ ohms, $f = 50$ Hz.

1. If the increment of f is $\Delta f = 2$ Hz, compute an approximation for $\Delta\tan\varphi$.
2. If the increment of R is $\Delta R = 5$ ohms, compute an approximation for $\Delta\tan\varphi$.

Exercise 15.
Calculate the differential logarithmic of :

1. $f(t) = \dfrac{r^{\alpha}(t)}{\theta^{\beta}(t)}$
2. $f(t) = \dfrac{t}{\sin t}$
3. $f(t) = r(t)\theta(t)$
4. $f(t) = r(t) + \theta(t)$

Exercise 16. The length and width of a rectangular metal plate are growing at the speed of $0,1\%$ by degree. What is the percentage change in degree of its range ?