FONCTIONS : Partie I - Domains, limits, Continuity

Objectifs

- **Know the definitions of limits**
- **Know how to calculate limits**
- **Understand the concepts of continuity and continuous extension**
- **Understand and know how to use $o()$ and $\sim$ notation**

1 Domains of definition.

In mathematics, a function is an operator that has a domain (starting set) and a codomain (target set). However, in practice, a function is often defined by a mathematical expression without explicitly stating its domain. The **domain of definition** of a function defined by an expression is the set of real numbers for which that expression is meaningful (meaning that all operations required to evaluate the expression are permissible). Some examples :

- If the expression contains a term $\dfrac{1}{u}$, one must ensure that $u \neq 0$.
- If the expression contains a term $\sqrt{u}$, one must ensure that $u \geqslant 0$.
- If the expression contains a term $\ln(u)$, one must ensure that $u > 0$.

Example 1.
Determine the domain of definition for the functions f, g, and h defined by

$$f(x) = \frac{x^2+2x-3}{x^2-1}, \quad g(x) = \frac{\ln(x)}{x-2}, \quad h(x) = \frac{\sqrt{x^3}}{x}.$$

2 Definition of the concept of a limit

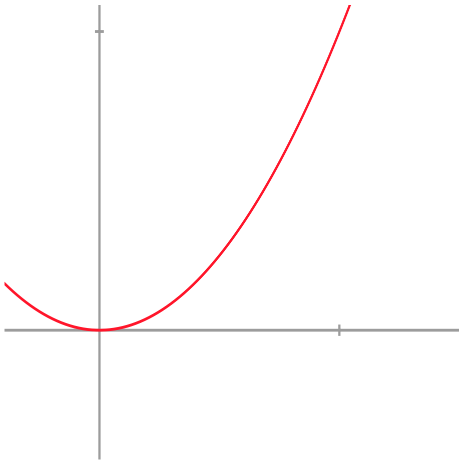
Throughout this section, we consider a function f from an interval I to $\mathbb{R}$.
$a \in \mathbb{R} \cup \{-\infty, +\infty\}$

2.1 Finite limit at a

Definition 1.
Let $l \in \mathbb{R}$.
Notation : $\lim_{x \to a} f(x) = l$ or $f(x) \to l$ as $x \to a$, read as : "f converges to l as x approaches a".

1. Case where $a \in \mathbb{R}$: we say that f has limit l at a if and only if :

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in I \backslash \{a\}, |x-a| \leqslant \alpha \Rightarrow |f(x)-l| \leqslant \varepsilon$$



Example 2. Write, using mathematical symbols, that $\lim_{x \to a} f(x) \neq l$.

2. Case where $a = +\infty$: we say that f has limit l at $+\infty$ if and only if :

$$\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in I, x \geqslant A \Rightarrow |f(x) - l| \leqslant \varepsilon$$

3. Case where $a = -\infty$: ? ?we say that f has limit l at $-\infty$ if and only if :

$$\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in I, x \leqslant A \Rightarrow |f(x) - l| \leqslant \varepsilon$$

Intuitively, the statement $\lim_{x \to a} f(x) = l$ has the following meaning : for any $\varepsilon > 0$, for $f(x)$ to be within a distance of $\leqslant \varepsilon$ from l, it suffices for x to be sufficiently close to a. However, one must be careful : this phrasing obscures the important idea behind the definition ?namely, that the neighborhood in question depends on ε.

Example 3. Show that if $f(x) \underset{x \to +\infty}{\longrightarrow} 1$, then for all sufficiently large x, $f(x) \geqslant 0$.

Theorem 1. *Uniqueness of the limit*
Let $f : I \to \mathbb{R}$ be a function, and let l and l' be two real numbers. Suppose that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = l'$. Then $l = l'$.

Remarque 1. The limit at a real number a is unique, but it is not necessarily equal to $f(a)$.

2.2 Infinite limit at a

In this paragraph, a is a finite or infinite endpoint of the interval I.

Definition 2.
Let $f : I \to \mathbb{R}$ be a function.

1. Case where $a \in \mathbb{R}$: we say that f admits $+\infty$ (respectively $-\infty$) as limit in a if and only if :
$$\forall B \in \mathbb{R}, \exists \alpha > 0, \forall x \in I \backslash \{a\}, |x - a| \leqslant \alpha \Rightarrow f(x) \geqslant B \text{ (respectively, f(x) } \leqslant \text{ B)}$$
2. Case where $a = +\infty$: we say that f admits $+\infty$ (respectively $-\infty$) as limit in $+\infty$ if and only if :
$$\forall B \in \mathbb{R}, \exists A \in \mathbb{R}, \forall x \in I, x \geqslant A \Rightarrow f(x) \geqslant B \text{ (respectively, f(x) } \leqslant \text{ B)}$$
3. Case where $a = -\infty$: we say that f admits $+\infty$ (respectively $-\infty$) as limit in $-\infty$ if and only if :
$$\forall B \in \mathbb{R}, \exists A \in \mathbb{R}, \forall x \in I, x \leqslant A \Rightarrow f(x) \geqslant B \text{ (respectively, f(x) } \leqslant \text{ B)}$$

We note : $\lim_{x \to a} f(x) = +\infty$ respectively $\lim_{x \to a} f(x) = -\infty$

Example 4.
Let f be a function defined on $\mathbb{R}$ and let a be a real number. Sketch a function f satisfying the following property :
$$\exists \alpha \in \mathbb{R} : \forall \varepsilon > 0, |x - a| < \alpha \Longrightarrow |f(x) - l| \leqslant \varepsilon$$

2.2.1 Right-hand limit, left-hand limit

Definition 3. *Left-hand limit*
Assume that x_0 is not the left endpoint of I. If we examine the values ? ?of $f(x)$ only for $x < x_0$ and $x \in I$, then we say we are studying the left-hand limit of f at x_0, denoted by : $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x)$ or $\lim\limits_{x \to x_0^-} f(x)$

Example 5. Determine $\lim\limits_{x \to 0^-} \frac{1}{x}$

Definition 4. *Right-hand limit*
Assume that x_0 is not the right endpoint of I. If we consider the values ? ?of $f(x)$ only for $x > x_0$ and $x \in I$, then we say that we are examining the right-hand limit of f at x_0, denoted as : $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x)$ or $\lim\limits_{x \to x_0^+} f(x)$

Example 6. Determine $\lim\limits_{x \to 0^+} \frac{1}{x}$

Property 1. We suppose that x_0 is an element of I distinct from the limits of I. Then f admits a limit at x_0 if and only if the limit on the right is equal to the limit on the left. The limit of f at x_0 is then equal to the limit on the right or left at x_0.

Remarque 2.
If x_0 is one of the limits of I then :

- If $I =]x_0; ...$ or $I = [x_0; ..$ then $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0^+} f(x)$.
- If $I = ...; x_0[$ or $I = ...; x_0]$ then $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0^-} f(x)$.

Example 7. We consider the function : $\delta : \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto \begin{cases} 1 \text{ si } x = 0 \\ \dfrac{|x|}{x} \text{ si } x \neq 0 \end{cases} \end{array}$

Does δ have a limit of 0 ?

Example 8. La fonction suivante admet-elle une limite en 0 ? $f : \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto \begin{cases} \sqrt{x}, \text{ si } x \geqslant 0 \\ -x \text{ si } x < 0 \end{cases} \end{array}$

3 Operations on limits.

In practice, one will almost never resort to the definition of a limit, but will instead use operations on limits. However, whenever an **indeterminate form (I.F.)** arises, one must examine the expression whose limit is being studied in greater detail. In the following, a denotes either a real number or infinity.

3.1 Limit of a sum

$\lim_{x \to a} (f(x) + g(x)) =$

$\lim_{x\to a} g(x) =$ \ $\lim_{x\to a} f(x) =$	$l \in \mathbb{R}$	$+\infty$	$-\infty$
$m \in \mathbb{R}$	$m + l$	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	**F.I.**
$-\infty$	$-\infty$	**F.I.**	$-\infty$

3.2 Limit of a product

$\lim_{x \to a} (f(x).g(x)) =$

$\lim_{x\to a} g(x) =$ \ $\lim_{x\to a} f(x) =$	$l > 0$	$l < 0$	0	$+\infty$	$-\infty$
$m > 0$	ml	ml	0	**F.I.**	**F.I.**
$m < 0$	ml	ml	0	**F.I.**	**F.I.**
0	0	0	0	**F.I.**	**F.I.**
$+\infty$	$+\infty$	$-\infty$	**F.I.**	$+\infty$	$-\infty$
$-\infty$	$-\infty$	$+\infty$	**F.I.**	$-\infty$	$+\infty$

3.3 Limit of a quotient

$\lim_{x \to a} \frac{f(x)}{g(x)} =$

$\lim_{x\to a} g(x) =$ \ $\lim_{x\to a} f(x) =$	$l > 0$	$l < 0$	0	$+\infty$	$-\infty$
$m > 0$	$\frac{l}{m}$	$\frac{l}{m}$	0	$+\infty$	$-\infty$
$m < 0$	$\frac{l}{m}$	$\frac{l}{m}$	0	$-\infty$	$+\infty$
0^+	$+\infty$	$-\infty$	**F.I.**	$+\infty$	$-\infty$
0^-	$-\infty$	$+\infty$	**F.I.**	$+\infty$	$+\infty$
$\pm\infty$	0	0	0	**F.I.**	**F.I.**

Example 9.
Give an example of an indeterminate form for each of the preceding cases.

Example 10. For each of the following examples, state whether or not one can directly conclude regarding the existence of a limit.

1. $\sqrt{x+2} - \sin(x)$ as $x \to 0$?

0. FI = Undertermine form, all limits are possible

2. $\dfrac{\sqrt{x^2+2}}{2x+7}$ as $x \to +\infty$?
3. $\dfrac{e^{-x}}{x^2}$ as $x \to 0$? As $x \to +\infty$?
4. $x - \sqrt{x^2+7}$ as $x \to +\infty$? As $x \to -\infty$?

3.4 Limit of a composite function

Definition 5.
Let f be a function defined on a set I and g be a function defined on $f(I)$. Then $g \circ f$ is the function defined on I by $g \circ f(x) = g(f(x))$ for all $x \in I$.

Example 11.

1. Determine $f \circ g$ and $g \circ f$ given $f(x) = x + 3$ and $g(x) = x^2 - 1$.
2. Determine two functions f and g such that $h = f \circ g$ with $h(x) = \sqrt{x^4 + 2x^2 + 1}$.

Theorem 2.
Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be such that $f(I) \subset J$ so that the composite $g \circ f$ makes sense.

1. If f has a limit (finite or infinite) b in a then b is an element or a limit of J
2. If in addition g has a limit (finite or infinite) l in b, then $g \circ f$ admits the limit l in a :

$$\begin{cases} f(x) \underset{x \to a}{\to} b \\ g(x) \underset{x \to b}{\to} l \end{cases} \Rightarrow g\left(f(x)\right) \underset{x \to a}{\to} l$$

Example 12.
Calculate $\lim\limits_{x \to +\infty} \ln\left(\dfrac{1}{x^2+1}\right)$

3.5 Negligible and equivalent functions.

3.5.1 Negligible functions, $o()$ notation

Definition 6.
Let a be a real number or an endpoint of an interval on which two functions f and g are defined. We say that f is **negligible compared to** g **near** a if

$$\frac{f(x)}{g(x)} \underset{x \to a}{\longrightarrow} 0.$$

This is denoted as "$f(x) = \underset{x \to a}{o}(g(x))$", "$f(x) = o(g(x))$ near a", or, if there is no ambiguity, $f = o(g)$. This notation is read as "f is little-o of g near a". It expresses the fact that $f(x)$ is "infinitesimally small" compared to $g(x)$ near a.

Example 13.

1. Show that $f(x) = x^2 - 3x + 1$ is negligible compared to $g(x) = x^3$ as $x \to +\infty$.
2. Show that $\sqrt{x} = o(x)$ as $x \to +\infty$, but that $x = o(\sqrt{x})$ as $x \to 0$.

Remarque 3. An important special case : $f(x) = o(1)$ as $x \to a$ if and only if $f(x) \to 0$ as $x \to a$. In particular, $f(x) = L + o(1) \iff f(x) \to L$ as $x \to a$.

We will see in a later chapter that this concept plays a role in the calculation of **comparative growth rates**, which are very useful for resolving **indeterminate forms** :

Proposition 1. For $\alpha, \beta > 0$, we have :

- in $+\infty$:
$$x^\alpha = o(x^\beta) \textit{ ssi } \alpha < \beta, \quad x^\alpha = o(e^x), \quad \ln x = o(x^\beta).$$
- in 0^+ :
$$x^\beta = o(x^\alpha) \textit{ ssi } \alpha < \beta, \quad \ln x = o\left(\frac{1}{x^\alpha}\right).$$

3.5.2 Equivalent functions

Definition 7. Two functions f and g are said to be equivalent in the neighborhood of a ?denoted $f \underset{a}{\sim} g$ or $f(x) \underset{x \to a}{\sim} g(x)$?if
$$\frac{f(x)}{g(x)} \underset{x \to a}{\longrightarrow} 1.$$
Equivalently, $f \sim g \iff f(x) = g(x)(1+o(1))$ as $x \to a$, or $f \sim g \iff f(x) = g(x) + o\left(g(x)\right).$

Example 14.

1. Show that $x^3 - 5x^2 + \frac{1}{x} \sim x^3$ as $x \to +\infty$.
2. Show that $x^3 - 5x^2 + \frac{1}{x} \sim \frac{1}{x}$ as $x \to 0$.
3. Show that $e^x \sim 1$ as $x \to 0$.

Remarque 4. It is evident from the definition that a function can NEVER be equivalent to 0.

To find a simple equivalent for an expression, one can factor out the dominant term (in the neighborhood of a) and verify that the remaining terms tend to 1.

Proposition 2. Some properties of asymptotic equivalents : in the neighborhood of a,

1. If $f \sim f_1$ and $g \sim g_1$, then $fg \sim f_1 g_1$.
2. If $f \sim f_1$ and $g \sim g_1$, then $\frac{f}{g} \sim \frac{f_1}{g_1}$.
3. If $f \sim g$ and $f \to L$ with $L \in \mathbb{R} \cup \{\pm\infty\}$, then $g \to L$.
4. A polynomial function is equivalent to its highest-degree term as $x \to +\infty$ or $x \to -\infty$.

This proposition is very useful in practice : it tells us that to find an equivalent for a quotient or a product, we simply need to find an equivalent for each of the terms. Furthermore, point 3 indicates that to find the limit of a function, we need only look at the limit of its equivalent, provided we have a simple equivalent available.

Example 15.

1. Determine the limit of $\frac{x^3 - 3x^2 + 12}{3x^3 + x^2 + x + 1}$ as $x \to +\infty$, and then as $x \to -\infty$.
2. Determine an equivalent as $x \to +\infty$ for $f(x) = (\sqrt{x} - 2) \cdot \frac{x^2 - 3x + 1}{3x^3 + 1}$, and then deduce its limit as $x \to +\infty$.

3.6 Limits involving square root functions

There are two standard examples to know :

- Indeterminate form as $x \to +\infty$ of the type $\dfrac{\sqrt{x+3}}{\sqrt{x-4}}$: factor out the equivalent term $(\sqrt{x})$ in both the numerator and the denominator.
- Indeterminate form of the type $\sqrt{x+3} - \sqrt{x+7}$: multiply and divide by the conjugate expression $\sqrt{x+3} + \sqrt{x+7}$.

Example 16.
Justify and calulate those limits

4 Limits and inequalities

4.1 Theorems

a is either a real number or infinity.

Theorem 3. *Passing from non-strict inequalities to the limit*
Let f and g be two functions from I to $\mathbb{R}$. Suppose that f and g both have finite limits l and m at a and that $f \leqslant g$ in a neighborhood of a ; then : $l \leqslant m$.

Example 17.
If f and g are two functions from I to $\mathbb{R}$ such that f and g both have finite limits l and m at a and $f < g$ in a neighborhood of a, is it true that $l < m$?

Theorem 4. *Squeeze Theorem*

Let f, g, h be three functions from I to $\mathbb{R}$. Suppose that : $\begin{cases} f(x) \underset{x\to a}{\longrightarrow} l \\ h(x) \underset{x\to a}{\longrightarrow} l \end{cases}$ and $f \leqslant g \leqslant h$ in a neighborhood of a ; then $g(x) \underset{x\to a}{\longrightarrow} l$.

Example 18. Determine the limit of $\dfrac{\cos x}{x}$ as $x \to +\infty$.

Remarque 5. (Case $l = 0$) In particular, the Squeeze Theorem implies that to show $f(x) \to 0$ as $x \to a$, it suffices to show that $0 \leqslant |f(x)| \leqslant g(x)$ for some function g such that $g(x) \to 0$ as $x \to a$. This allows us to use the triangle inequality to establish upper bounds on the absolute value in order to show that a function tends to 0.

Example 19. Show that $\dfrac{1 + 3\cos(x) - \sqrt{x}}{x} \longrightarrow 0$ as $x \to +\infty$.

Property 2.

1. Let f, g be two functions from I to $\mathbb{R}$. Suppose $f(x) \to +\infty$ as $x \to a$ and $f \leqslant g$ in a neighborhood of a ; then $g(x) \to +\infty$ as $x \to a$.
2. Let f, g be two functions from I to $\mathbb{R}$. Suppose $g(x) \to -\infty$ as $x \to a$ and $f \leqslant g$ in a neighborhood of a ; then $f(x) \to -\infty$ as $x \to a$.

Example 20.
Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto x + \cos x$. What are $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$?

5 Monotone Limit Theorem

Theorem 5.
Let f be a function defined on an interval $I =]a, b[$, with a and b in $\mathbb{R} \cup \{-\infty, +\infty\}$. If f is monotonic on I, then it has a finite or infinite limit at a and at b.

Example 21.

1. What can be said about the limit of a function f in the following cases :
   (a) f is increasing ?
   (b) f is bounded above ?
2. What sufficient condition(s) must f satisfy for it to have a finite limit ? Are these conditions necessary ?

6 Continuity

6.1 Continuous function

Here, we study the limit of a function at a point where it is defined.

Definition 8.
Let $f : I \to \mathbb{R}$ be a function.

1. Let $x_0 \in I$. f is continuous at x_0 means that :
   - f has a finite limit at x_0
   - $\lim\limits_{x \to x_0} f(x) = f(x_0)$

   If either of these two conditions is not met, f is said to be discontinuous at x_0.
2. f is said to be continuous on I if and only if it is continuous at every point x_0 in I.

Example 22.
Is the function δ defined in Example 7 continuous at 0 ?

Property 3.
If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Example 23.
Let f be the function defined by $f(x) = \ln(\sqrt{x} + 1)$. Is f continuous at 0 ?

Property 4.
Let f be a function defined via operations on standard functions (sin, cos, rational functions, etc.). Then f is continuous on its domain.

Remarque 6.
To study the continuity of a function on a set, we apply the previous property to the subset where f is defined via operations on standard functions, and we use Definition 8 for the remaining real numbers.

Example 24.
Let f be the function defined on $\mathbb{R}$ by :

$$\begin{cases} f(0) = 2 \\ f(x) = \dfrac{\sqrt{1+x^2}}{x^2} \quad \text{if } x \neq 0 \end{cases}$$

Is the function f continuous on $\mathbb{R}$?

6.2 Continuous extension

Definition 9.
Let f be a function defined on an open interval I, and let a be an endpoint of I. If f has a limit l at a, we say that f can be **extended by continuity at** a by setting $f(a) = l$.

Example 25.
Let f be the function defined by $f(x) = \dfrac{x^2+2x-3}{x^2-1}$. State the domain of this function. Can f be extended by continuity at $a = 1$?

Exercices

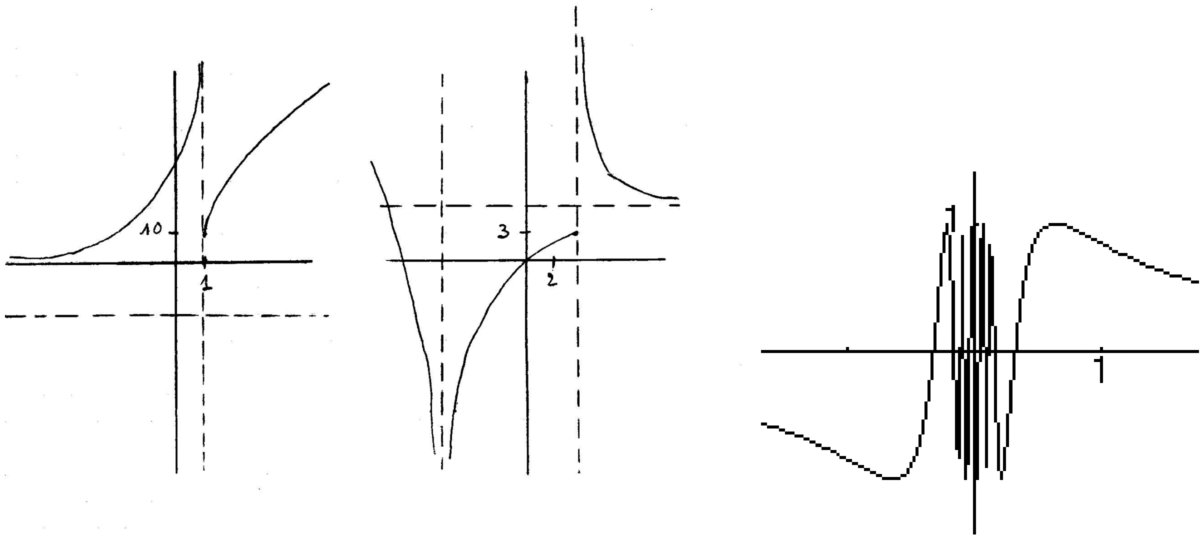
Exercise 1.
Give the domain of definition of the following functions :

1. $f(x) = \dfrac{1}{x+2}$;
2. $f(x) = \dfrac{\sin x}{x}$
3. $f(x) = \sqrt{-2x+3}$
4. $f(x) = \ln(x^3 - 4x^2 + x)$

Exercise 2.
In each of the following cases, sketch a possible shape of the curve given that :

1. $\lim\limits_{2^+} f = 3$, $\lim\limits_{2^-} f = -\infty$, $\lim\limits_{+\infty} f = 5$ and $\lim\limits_{-\infty} f = +\infty$.
2. $\lim\limits_{-1^+} f = -\infty$, $\lim\limits_{-1^-} f = 0$, $\lim\limits_{-\infty} f = 0$ and $\lim\limits_{+\infty} f = -\infty$.

Exercise 3.
In the following examples, conjecture the limits of f. Explain why these are merely conjectures.



Exercise 4. For each of the following mathematical statements, give the meaning in terms of limits and then write the negation of the statement.

1. $\forall \varepsilon > 0$, $\exists A \in \mathbb{R}$, $\forall x \in I\ x \geqslant A \implies |f(x)+2| \leqslant \varepsilon$.
2. $\forall \varepsilon > 0$, $\exists \alpha > 0$, $\forall x \neq 0\ |x| \leqslant \alpha \implies |f(x)+3| \leqslant \varepsilon$.
3. $\forall A \in \mathbb{R}$, $\exists M \in \mathbb{R}$, $|x| > M \implies |x| < A$.

Exercise 5.
Write, using mathematical symbols, that a function does not have a finite limit at $+\infty$.

Exercise 6. 1. Study the limits in $+\infty$:

$$a)\ \frac{3x+7}{6x-13} + \frac{14x^2+2}{7x^2+31x-4} - 5 + \frac{1}{3x+4}, \qquad b)\ \frac{x^3}{x-2} - \frac{x^2}{x-1}.$$

2. Study the limits in $+\infty$, $-\infty$ and 0

$$\frac{e^x + 1 + \frac{1}{x}}{e^x + 4}.$$

Exercise 7.
For $R, C, \omega > 0$ and $t \in \mathbb{R}$, determine the limits (if they exist) of

$$\frac{CR - \omega t^2}{CRt + \frac{1}{\omega t}}$$

as $t \to +\infty$, $t \to -\infty$, and $t \to 0$.

Exercise 8. Are the following statements true or false ? (Justify your answers.)

1. $\sin(x) = o(1)$ as $x \to 0$.
2. $x^3 + x = o(x^4)$ as $x \to 0$.
3. $x^3 + x - \ln(x) = o(x^4)$ as $x \to +\infty$.
4. $x + \sin(x) \sim x$ as $x \to -\infty$.
5. $e^{4x+4} = o(e^{(x^2)})$ as $x \to +\infty$.

Exercise 9. Determine a simple equivalent for each of the following functions :

1. $x^3 - x\sqrt{x} + e^{-x^2} + (\ln(x))^2$ as $x \to +\infty$.
2. $\dfrac{2x + x^4 + 3x^7}{x^3 + 3}$ as $x \to 0$, and as $x \to +\infty$.
3. $x + e^x$ as $x \to +\infty$; as $x \to -\infty$; as $x \to 0$.
4. $\dfrac{x^2 + 3x}{2 + x} - x$ as $x \to +\infty$.
5. $\sqrt{x^2 + 10}$ as $x \to +\infty$; as $x \to -\infty$.

Exercise 10. Calculate the following limits when they exist

$a)$ $\displaystyle\lim_{x\to 0} \frac{x^2 + 2\,|x|}{x}$ $b)$ $\displaystyle\lim_{x\to -\infty} \frac{x^2 + 2\,|x|}{x}$ $c)$ $\displaystyle\lim_{x\to 2} \frac{x^2 - 4}{x^2 - 3\,x + 2}$

$d)$ $\displaystyle\lim_{x\to \pi} \frac{1 - \cos^2 x}{1 + \cos x}$ $e)$ $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1+x^2}}{x}$ $f)$ $\displaystyle\lim_{x\to +\infty} \sqrt{x+5} - \sqrt{x-3}$

Exercise 11.
Determine the limit of the following functions :

1. $\displaystyle\lim_{x\to +\infty} -2x - 3\cos x$
2. $\displaystyle\lim_{x\to -\infty} x + \sin x$
3. $\displaystyle\lim_{x\to +\infty} \frac{x + \cos x + 1}{x - \sin x}$
4. $\displaystyle\lim_{x\to +\infty} x + 2 + \left(\frac{\sin(x) - 3 + \ln(x)}{4 + \sqrt{x} + \ln(x)}\right)^2$

Exercise 12.
Are the following continuous at a ?

1. $f(x) = \dfrac{x}{\sqrt{|x|}}$ pour $x \neq 0$ et $f(0) = 1$, en $a = 0$.
2. $f(x) = \dfrac{x^2 + x - 2}{x^2 + 3x - 4}$ pour $x \neq 1$ et $x \neq -4$ et $f(1) = 10$ en $a = 1$.

Exercise 13.
Investigate whether the given functions can be extended to the specified points :

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$
2. $f(x) = \dfrac{x^2 - 9}{\sqrt{x + 3}}$ at $x = -3$
3. $f(x) = \dfrac{\sqrt{3x^2 + 1} - 2}{x - 1}$ at $x = 1$
4. $f(x) = \ln\left(1 + \dfrac{\frac{1}{|x|} + x}{\frac{1}{|x|} + 3}\right)$ at $x = 0$.
5. $f(x) = \exp\left(\dfrac{x^2 - 6x + 8}{(x - 2)^3}\right)$ at $x = 2$.

Exercise 14.
Let f be the function defined by $f(x) = (-1)^{\lfloor x \rfloor}(x - \lfloor x \rfloor)$, where $\lfloor x \rfloor$ represents the integer part of x (i.e., the greatest integer less than or equal to x). Study the continuity of f.