Study of functions and usual functions

Objectifs
- **To b able to study a real-valued function of one variable**
- **To know usual functions**

1 To study a real-valued function of one variable

1.1 Parity

Definition 1. Even Function
Let f be a real-valued function from I to $\mathbb{R}$, of one variable. f is an even function if :
- I is symmetric with respect to the origin .
- $\forall x \in I,\ f(-x) = f(x)$

Example 1.
Are those functions even on their domain of definition ?
$f(x) = \dfrac{x^2 + \cos x + 1}{-|x| - 1}$; $g(x) = 2x + 2$

Property 1.
A function is even if and only if its graph face is symmetric with respect to the y-axis, meaning that its graph remains unchanged after reflection about the y-axis.

Example 2.
Check this property with a graph.

Definition 2. Odd Function
Let f be a real-valued function from I to $\mathbb{R}$, of one variable. f is an odd function if :
- I is symmetric with respect to the origin .
- $\forall x \in I,\ f(-x) = -f(x)$

Example 3.
Are those functions odd on their domain of definition ?
$f(x) = \dfrac{-x^3 + \sin x}{x^2 + 1}$; $g(x) = 2x + 2$

Property 2.
A function is odd if and only if its graph has rotational symmetry with respect to the origin, meaning that its graph remains unchanged after rotation of 180 degrees about the origin.

Example 4.
Check this property with a graph.

1.2 Asymptotic directions

On a une branche infinie si x or $f(x)$ tends to $+\infty$ or $-\infty$.

- first case : $\lim_{x\to x_0} f(x) = \pm\infty$: vertical asymptote $x = x_0$
- second case : $\lim_{x\to\pm\infty} f(x) = y_0$: horizontal asymptote $y = y_0$
- third case : $\lim_{x\to\pm\infty} f(x) = \pm\infty$. We study $\lim_{x\to\pm\infty} \frac{f(x)}{x}$
  1. If $\lim_{x\to\pm\infty} \frac{f(x)}{x} = \pm\infty$: asymptotic curve of asymptotic direction Oy (example : the function exp)
  2. If $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$: branche parabolique de direction asymptotique Ox (exemple : la fonction $\ln x$)
  3. If $\lim_{x\to\pm\infty} \frac{f(x)}{x} = a \in \mathbb{R}^*$: we study $\lim_{x\to\pm\infty} f(x) - ax$:
     (a) If $\lim_{x\to\pm\infty} f(x) - ax = b \in \mathbb{R}$: oblique asymptote $y = ax + b$
     (b) If $\lim_{x\to\pm\infty} f(x) - ax = \pm\infty$: asymptotic curve of asymptotic direction $y = ax$

Example 5.
Let's study asymptotic directions for f defined by $f(x) = \frac{2-3x^2}{x+3}$

1.3 Convexity, concavity for functions of diffrentiability class $\mathcal{C}^2$

In this section, let f be a function of differentiability class $\mathcal{C}^2$ on an open interval I.

Definition 3.
f is said convex on I if $\forall x \in I, f''(x) \geqslant 0$. Instead of "convex", one often says "curved upwards".

Example 6.
Let f be a convex function on an interval I.

1. What can you say about the derivative of f ?
2. Let's sketch the graph of a convex function.

Definition 4.
f is said concave I if $-f$ is convex on I. Instead of "concave", one often says "curved downwards".

Example 7.
Sketch the graph of a concave function, using the previous definition.

Definition 5. Let $x_0 \in I$. f has an **inflection point** at x_0, if and only if the sign of $f''(x)$ changes at x_0.

Sketch an inflection point.

1.4 General method for studying a function

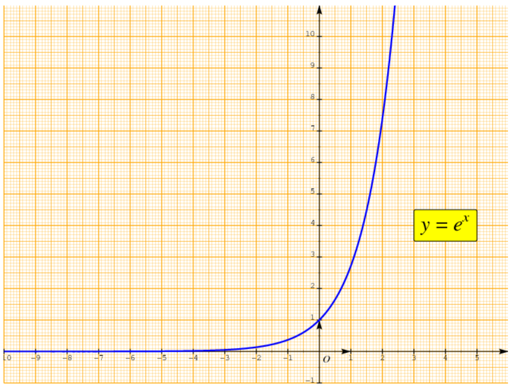
1. Domain of definition, , oddness, evenness, periodicity (if possible you reduce the domain to study).
2. Continuity, extension by continuity, differentiability.
3. Sum up the variations of the function : compute the derivative, study its monotony and let's denote its stationnary points (where f' vanishes) and forbidden points (where f' is not defined).
4. Tangentes :
   (a) If the derivative function vanishes : horizontal tangent line
   (b) If the derivative function or the difference quotient tends to $+\infty$ or $-\infty$: vertical tangente line, non differentiability point.
   (c) half-tangente lines : get while studying f on the right-hand or left-hand of x_0
5. Limits at endpoints.
6. Study asymptotic directions.
7. Give the full variation table of f.
8. Concavity, convexity, inflection point.
9. Sketch the graph

2 Exponential, logarithmic and power functions

2.1 Exponential function

Definition 6. There exists a unique function from $\mathbb{R}$ to $\mathbb{R}$, called the exponential function appelée, denoted exp, differentiable on $\mathbb{R}$ such that :

$$\begin{cases} \exp(0) = 1 \\ \exp'(x) = \exp(x), \ \forall x \in \mathbb{R} \end{cases} \quad \textbf{Fonctionnal equation}$$



By definition, exp is continuous and differentiable on $\mathbb{R}$.

$$\forall x, y \in \mathbb{R} \ : \begin{cases} \exp(x+y) = \exp(x).\exp(y) \\ \exp(-x) = \dfrac{1}{\exp(x)} \end{cases}$$

We set : $e = \exp(1)$ and denote $e^x = \exp(x)$.

Variations

$\forall x \in \mathbb{R}, \exp(x) > 0$ thus $\exp' = \exp$, so exp is strictly increasing.

Limits at endpoints

$$\lim_{x \to +\infty} e^x = +\infty \qquad \lim_{x \to -\infty} e^x = 0$$

The graph of exp has an horizontal asymptote $-\infty$ of equation $y = 0$ this is the X-axis.

Comparative growths

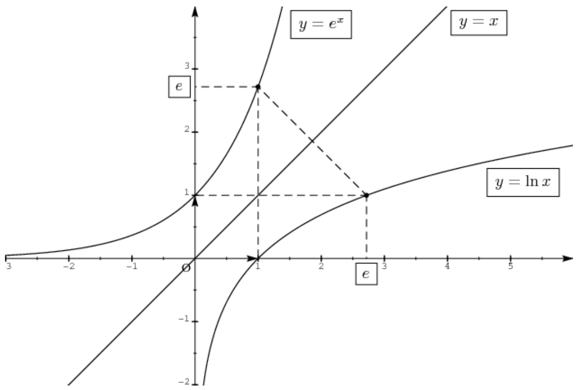
$$\begin{cases} \forall n \in \mathbb{R}^*, \lim_{x \to +\infty} \frac{e^x}{x^n} = +\infty \\ \forall n \in \mathbb{R}^*, \lim_{x \to -\infty} |x|^n e^x = 0 \end{cases}$$

Limits to know by heart

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

2.2 Logarithmic function

Definition 7. The logarithmic function, denoted by ln, is the inverse function of the exponential exp as exp is bijective from $\mathbb{R}$ to $]0; +\infty[$. The logarithmic function ln is defined from $]0; +\infty[$ to $\mathbb{R}$.



Variations
ln has the same variations of exp thus ln is continuous, differentiable and strcitly increasing on $]0; +\infty[$.
Derivative

$$\forall x > 0, \ln'(x) = \frac{1}{\exp'(\ln x)} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Limits at endpoints

$$\lim_{x \to 0^+} \ln x = -\infty \qquad \lim_{x \to +\infty} \ln x = +\infty$$

The graph of the logarithmic function ln has a vertical asymptote at 0.

Fonctionnal equation

$$\forall x, y > 0 \ : \begin{cases} \ln(xy) = \ln(x) + \ln(y) \\ \ln \frac{1}{x} = -\ln x \end{cases}$$

Comparative growths

$$\begin{cases} \forall n \in \mathbb{N}^*, \lim_{x \to +\infty} \frac{\ln x}{x^n} = 0 \\ \forall n \in \mathbb{N}^*, \lim_{x \to 0^+} x^n \ln x = 0 \end{cases}$$

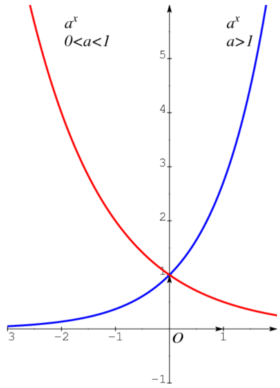
Limit to know by heart

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

Example 8. Let's study $f : x \mapsto \ln(1 + e^x)$

2.3 Exponential and logarithmic function with base a

2.3.1 Exponenetial function with base a

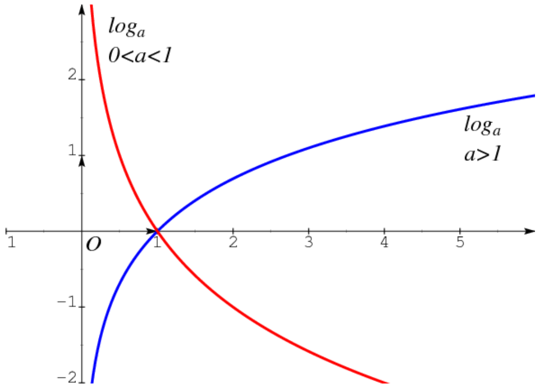


Definition 8. Let $a > 0$ be a strictly positive real number. For all $x \in \mathbb{R}$, on définit l'exponentielle de base a par :

$$a^x = \exp(x \ln a) = e^{x \ln a}$$

Ainsi, l'étude d'une exponentielle de base a se ramène à celle d'une exponentielle classique du type $e^{\alpha x}$.

2.3.2 Logarithmic function with base a



Let $a > 0$ be a strictly positive real number and $a \neq 1$. For all $x > 0$, the logarithmic function with base a is defined by :

$$\log_a(x) = \frac{\ln x}{\ln a}$$

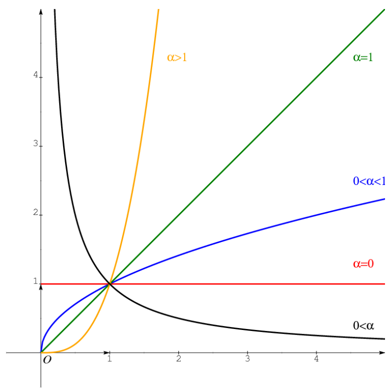
To study the logarithmic function with base a is equivalent to study the logarithmic function with a multiplicative factor.

2.3.3 Fonction logarithme décimal

The logarithm with base 10 is called the common logarithmic function, and denoted by This function is the inverse function of the exponential function of base 10 : $x \to 10^x$. Those functions are used when we work with powers of 10.

Example 9. For instance, in chemistery, we get the formula : $[H^+] = 10^{-PH}$. Let's deduce PH in function of the concentration $[H^+]$.

2.4 Power Functions



For $\alpha \in \mathbb{R}$, we define :

$$f_\alpha : \begin{pmatrix} \mathbb{R}_+^* \to \mathbb{R} \\ x \mapsto x^\alpha \end{pmatrix}$$

with $x^\alpha = e^{\alpha \ln x}$.
So , again , it comes down to study a classic exponential , except in the case where $alpha$ is an integer (conventional power function), and in the case of a negative integer (inverse function of a conventional power function) , rational and we have $x^{\ fracpq} = \ sqrt[q]x^p$.

2.5 Comparative growths

Let's consider $\alpha > 0$ and $a, b > 1$, we get :

$$\lim_{x\to+\infty} \frac{a^x}{x^\alpha} = +\infty \qquad \lim_{x\to+\infty} \frac{\log_b x}{x^\alpha} = 0 \qquad \lim_{x\to 0} x^\alpha \log_b x = 0$$

We summarize it : en $+\infty$:

$$\log_b x \ll x^\alpha \ll a^x$$

3 Hyperbolic Functions

3.1 The hyperbolic cosine and sine

Definition 9. The hyperbolic cosine and sine are functions from $\mathbb{R}$ to $\mathbb{R}$ defined by :

$$\begin{cases} \text{ch} : x \mapsto \dfrac{e^x + e^{-x}}{2} \\ \text{sh} : x \mapsto \dfrac{e^x - e^{-x}}{2} \end{cases}$$

Continuity, Differentiability
ch and sh are continuous and differentiable on $\mathbb{R}$

Let's study properties of ch and sh.

Oddness, Evenness

- The hyperbolic cosine ch is ....
- The hyperbolic sine sh is ....

Derivative

$$\forall x \in \mathbb{R}, \begin{cases} \text{ch}'(x) = \\ \text{sh}'(x) = \end{cases}$$

Variations Table

x	
$\text{ch}\, x$	

x	
$\text{sh}\, x$	

Sign
For all $x \in \mathbb{R}$, we get $\text{ch}(x)$ ....
$\text{sh}(x)$ changes sign at ....

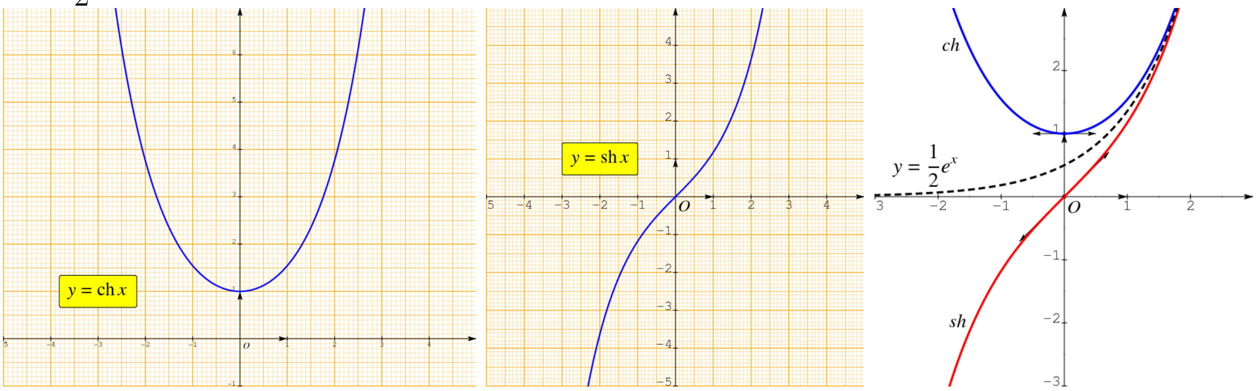
Hyperbolic Trigonometry
There exists many formulas between ch and sh, this one is **fundamental** :

$$\forall x \in \mathbb{R}, \text{ch}^2 x - \text{sh}^2 x = 1$$

Example 10. Prove the previous equality.

Comparaison

As $\operatorname{ch} x - \operatorname{sh} x = e^{-x}$ for all $x \in \mathbb{R}$, we get : $\lim\limits_{x \to +\infty} \operatorname{ch} x - \operatorname{sh} x = 0$. The curves of ch, sh and $x \mapsto \frac{1}{2}e^x$ are said asymptotic.



3.2 The hyperbolic tangent

Definition 10. The hyperbolic tangent, denoted by th, is defined on $\mathbb{R}$ by :

$$\forall x \in \mathbb{R}, \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

Continuity Differentiability

The functin th is continuous and differentiable on $\mathbb{R}$.

Oddness, Evenness

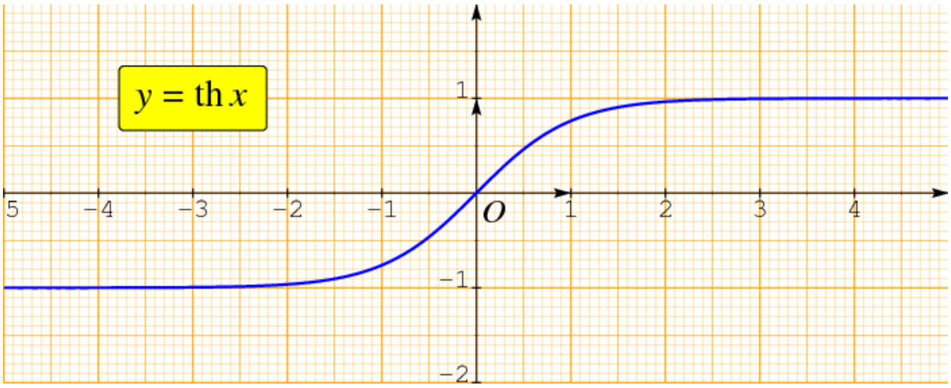
This function th is ....

Derivative

$$\forall x \in \mathbb{R}, \operatorname{th}'(x) = \text{..........................}$$

Variations table

x	
th x	



Example 11. Prove that : $\forall x \in \mathbb{R}, \operatorname{ch}(2x) = \operatorname{ch}^2 x + \operatorname{sh}^2 x$ and $\operatorname{sh}(2x) = 2 \operatorname{ch} x . \operatorname{sh} x$

4 Geometrical transformations of functions

4.1 Introduction

Let's try

1. Sketch the graph of the exponential function.
2. Sketch also the graph of the following functions : $-\exp(x)$; $\exp(x) + 1$; $\exp(x) - 1$; $2\exp(x)$.
3. Let's find geometrical tranformations used to move those graphs.
4. Same questions for the functions $\exp(-x)$; $\exp(x+1)$; $\exp(x-1)$; $\exp(2x)$.

Connect each formula up to a geometric transformation.

1. $-f(x)$;
2. $f(x) + 1$;
3. $f(x) - 1$;
4. $2f(x)$;
5. $f(-x)$;
6. $f(x+1)$;
7. $f(x-1)$;
8. $f(2x)$.

1. up translation ;
2. down translation ;
3. left translation ;
4. right translation ;
5. reflection symmetry of x-axis ;
6. reflection symmetry of y-axis ;
7. vertical dilation by factor 2 ;
8. horizontal dilation by factor 1/2.

1. Which formula is associated to a homothety of center the origin and of ration 2 ?
2. rotational symmetry with respect to the origin ?

4.2 Conclusion

Geometrical transformations can also be used to quickly move and resize graphs of functions. Let f and g be two real-valued functions of one variable, defined respectively on D_f and D_g, let's consider their graphs C_f and C_g in the orthonormal plane $(O, \vec{i}, \vec{j})$.

4.2.1 Translations

Theorem 1 (Vertical Translation)**.**
If $g(x) = f(x) + k$ with $k \in \mathbb{R}$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under the vertical translation of vector $k\vec{j}$.

Theorem 2 (Horizontal Translations)**.**
If $g(x) = f(x + k)$ with $k \in \mathbb{R}$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under the horizontal translation of vector $-k\vec{i}$.

4.2.2 Symmetries

Theorem 3 (Reflection symmetry of x-axis)**.**
If $g(x) = -f(x)$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under the reflection symmetry across the x-axis.

Theorem 4 (Reflection symmetry of y-axis)**.**
If $g(x) = f(-x)$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under the reflection symmetry accross the y-axis.

Theorem 5 (Point reflection (central inversion) at the origin O)**.**
If $g(x) = -f(-x)$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under the point reflection at the origin O.

4.2.3 Dilations

Theorem 6 (Vertical Dilation)**.**
If $g(x) = k.f(x)$ with $k > 0$
Then C_g is the image of C_f under a vertical dilation by a factor of k.

Theorem 7 (Horizontal Dilation)**.**
If $g(x) = f(k.x)$ with $k > 0$
Then C_g is the image of C_f under an horizontal dilation by a factor of $\frac{1}{k}$.

Theorem 8 (Homothety of center O and of ratio k)**.**
If $g(x) = k.f(\frac{1}{k}.x)$ with $k > 0$
Then C_g is the image of C_f under the homothety of center O of ratio k.

5 Trigonometric functions

5.1 Periodic functions

Definition 11.
Let T be a real, and f a real-valued function defined on I. f is a T peridoc function if :
- $\forall x \in I,\ x+T \in I.$
- $\forall x \in I,\ f(x+T) = f(x)$

Example 12.
Let's consider f defined by $f(t) = \cos(\omega t)$ for $t \in \mathbb{R}$. Prove thar f is a periodic function with period $\frac{2\pi}{\omega}$. ω is called pulsation in physics.

Proposition 1.
Let a, b, and ω be real numbers. Then there exists three real numbers ϕ, ϕ' and A such that :

$$\text{for all real}t : a\cos(\omega t) + b\sin(\omega t) = A\sin(\omega t + \phi) = A\cos(\omega t + \phi')$$

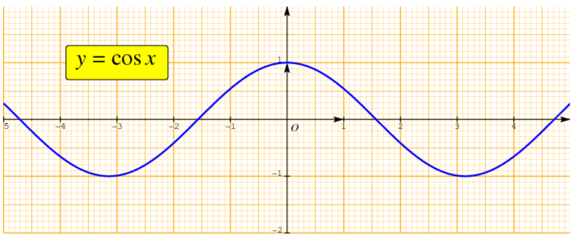
We get : $A = \sqrt{a^2+b^2}$, $\tan\phi = \frac{a}{b}$ and $\tan\phi' = -\frac{b}{a}$

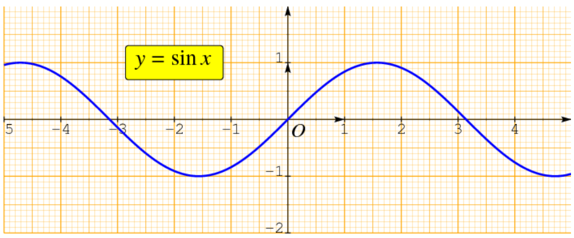
The previous property is useful in physics : the sum of two sinusoidal signals of the same pulse (and therefore same period) is a sinusoidal signal of same pulse (and therefore same period) with a phase difference of *phi* .

Example 13.
Let's prove the previous property, then let's write the expression $\cos 2x + \sin 2x$ as $A\sin(\omega x+\phi)$.

5.2 The cosine and sine functions

5.2.1 Graphs





5.2.2 Formulas

$\cos(-x) = \cos(x)$	$\sin(-x) = -\sin(x)$
$\cos(\pi - x) = -\cos(x)$	$\sin(\pi - x) = \sin(x)$
$\cos(\pi + x) = -\cos(x)$	$\sin(\pi + x) = -\sin(x)$
$\cos(\frac{\pi}{2} - x) = \sin(x)$	$\sin(\frac{\pi}{2} - x) = \cos(x)$
$\cos(\frac{\pi}{2} + x) = -\sin(x)$	$\sin(\frac{\pi}{2} + x) = \cos(x)$

Few fundamental values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	

Addition formulas
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\sin(a+b) = \cos(a)\sin(b) + \cos(b)\sin(a)$

Linéarisation
- $\cos^2(a) = \dfrac{1+\cos(2a)}{2}$
- $\sin^2(a) = \dfrac{1-\cos(2a)}{2}$
- $\sin(a)\cos(a) = \dfrac{1}{2}\sin(2a)$

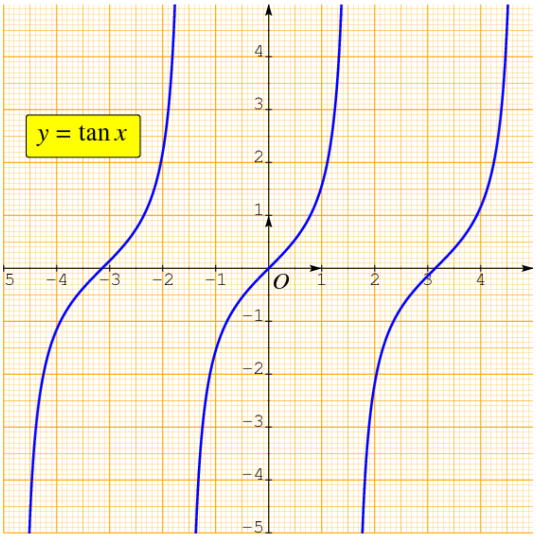
Solving an equation $\cos(x) = \cos(a)$

$$\cos(a) = \cos(b) \Leftrightarrow \begin{cases} a = b + 2k\pi, \quad k \in \mathbb{Z} \quad (1) \\ \text{ou} \\ a = -b + 2k'\pi, \quad k' \in \mathbb{Z} \quad (2) \end{cases}$$

Solving an equation $\sin(x) = \sin(a)$

$$\sin(a) = \sin(b) \Leftrightarrow \begin{cases} a = b + 2k\pi, \quad k \in \mathbb{Z} \quad (1) \\ \text{ou} \\ a = \pi - b + 2k'\pi, \quad k' \in \mathbb{Z} \quad (2) \end{cases}$$

5.3 The tangent function



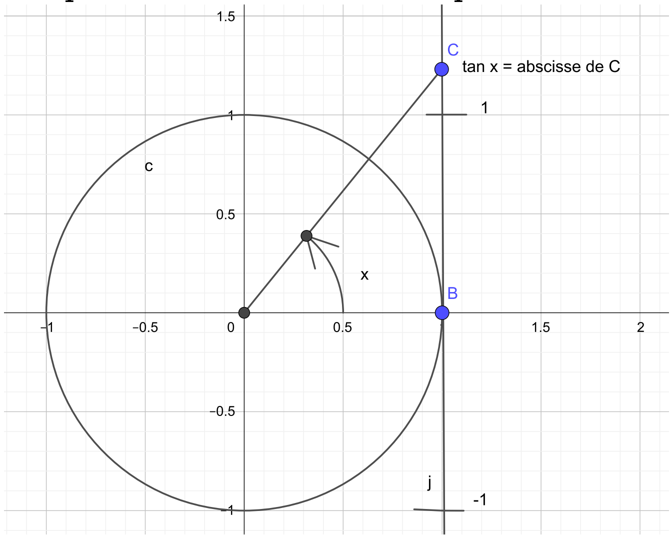
Definition 12.
The tangent function is defined by $\tan x = \dfrac{\sin x}{\cos x}$ on $\mathbb{R} - \{\dfrac{\pi}{2} + k\pi, \ k \in \mathbb{Z}\}$

Proposition 2.
- The tangent function is π periodic and odd function.
- $\tan'(x) = 1 + \tan^2 x = \dfrac{1}{\cos^2 x}$
- $\lim\limits_{\frac{\pi}{2}^-} \tan x = +\infty$

Solving an equation $\tan(x) = \tan(a)$

$$\tan(a) = \tan(b) \Leftrightarrow a = b + k\pi,\ k \in \mathbb{Z}$$

Graphical representation



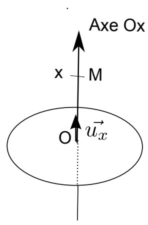
Example 14. Let's study $f : x \mapsto \tan x - \dfrac{1}{\tan x}$

Workout

Exercise 1.
Let's study those functions :

1. $U(r) = \dfrac{r^2 - R}{r + R}$ with R is a non zero constant.
2. $I(p) = \sqrt{ap - 1} - p$ with a a strictly positive real number.

Exercise 2.
In electromagnetism , you will learn about, the field and the electric potential created by a disk of radius R , flat metal (and thickness) uniformly electrified with an areal density of constant charge $sigma$. You will find that at a point M of abscissa x , the electric potential is :



$$V(x) = V(M) = \frac{\sigma}{2\varepsilon_0}(\sqrt{R^2 + x^2} - \sqrt{x^2})$$

1. Study V. Is V of differentiability class C^1 on $\mathbb{R}$?
2. We define $\vec{E}$, the electric field at the point M, by

$$\vec{E} = -\frac{dV}{dx}\vec{u_x}$$

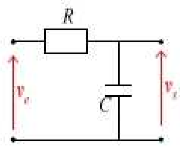
   Using the study of V, evaluate the limit of E at 0 and at $+\infty$.
3. Are those results consistent with the physical phenomenom ?

Exercise 3.
Let's study

1. $f(x) = xe^{\frac{1}{x}}$

2. $f : x \mapsto x^{\frac{1}{x}}$

Exercise 4. Bode's diagramm



Electronic filters are used to filter certain frequencies. To study these filters we define a transfer function which is equal to the quotient of the complex output voltage on the complex input voltage. For the circuit shown against, we get $H(j\omega) = \dfrac{1}{1 + j\frac{\omega}{\omega_0}}$ with $j^2 = -1$ et $\omega_0 = \dfrac{1}{RC}$. We then define the gain, in décibel, by $G_B = 20 \log |H(j\omega)|$.

1. Express in function of ω, $G_B(\omega)$.
2. Let g be the function defined $g(\omega) = -20 \log(\dfrac{\omega}{\omega_0})$.
   Prove that $\lim\limits_{\omega \to +\infty} G_B(\omega) - g(\omega) = 0$. What can you deduce about the curves of G_B and g ?
3. Let's study the variations of G_B.
4. We define : $X = \log \dfrac{\omega}{\omega_0}$ for $\omega > 0$, and K the function defined by $G_B(\omega) = K(X)$.
   Without express K in function of X :
   (a) Let's study the variations of K.
   (b) Determine the equations of the asymptotes of the representation of K as X tends to $-\infty$ and tends to $+\infty$.
   (c) Détermine the coordinates of the point $A(X_0, Y_0)$ of intersection between the previous lines.
   (d) Compute $K(X_0)$.
   (e) Sketch the trend of K, and its asymptotes in an orthogonal plane.
5. On the X-axis, we match each integer value of X with $\omega = k\omega_0$ such that $\log \dfrac{\omega}{\omega_0} = X$, thus we get $G(\omega)$ on the Y-axis.
   Thus we get the representation of G_B in a semi-log plot (or semi-log graph).
   The frequencies go from a few tens of hertz to several hundred , this will represent G_B.
6. ω_0 is called the cutoff frequency. Do you understand this name ?
7. We define the gain of a power (in decibel) by P par $G_B = 10 \log \dfrac{P_S}{P_E}$. One can also define the cut-off frequency for which the output power is equal to half of the input power . Prove that we refind $G_B(\omega_0)$

Exercise 5 (A familly of functions)**.**
Our goal is to study, for all natural number n distinct from 0, the functions f_n defined by

$$f(x) = x - n - n\frac{\ln x}{x}$$

We denote C_n the curve of f_n in an orthogonal plane.
A. Let's study f_n

1. Give the domain of definition of f_n.

2. let's define g_n on $]0;+\infty[$, by $g_n(x) = x^2 - n + n\ln x$
   (a) Prove that there exists a unique α_n such that $g(\alpha_n) = 0$, and that $\alpha_n \in [1;3]$. Let's compute α_1.
   (b) Compute f_n' and deduce variations of f_n.
   (c) Prove that $f_n(\alpha_n) = 2\alpha_n - n - \dfrac{n}{\alpha_n}$ and deduce the limit of $f_n(\alpha_n)$.
3. Study asymptotic directions of f_n, study the relative position of C_n with respect to its asymptotes on $]0;+\infty[$.

B. Relative positions of the graphs C_n

Soit $d(x) = f_n(x) - f_{n+1}(x)$.

1. Study the variations of d, evaluate its limits at 0 and at $+\infty$.
2. Deduce that the equation $d(x) = 0$ has a unique solution β and that $\beta \in [0;1]$.
3. Let's deduce the relative position of C_n and C_{n+1}.
4. Sketch on the same graph the trend of C_n and C_{n+1}.

Exercise 6.

Study those functions

1. $h : x \mapsto \operatorname{ch}\left(\dfrac{2x-1}{x+1}\right)$
2. $i : x \mapsto \ln(1 + \operatorname{th} x)$

Exercise 7.

Express $\operatorname{ch}(x)$ using $\operatorname{sh}(x)$ and $\operatorname{sh}(2x)$.

Simplify $u_n = \displaystyle\prod_{p=1}^{n} \operatorname{ch}\left(\frac{1}{2^p}\right)$, and deduce $\displaystyle\lim_{n\to+\infty} u_n$.

Exercise 8.

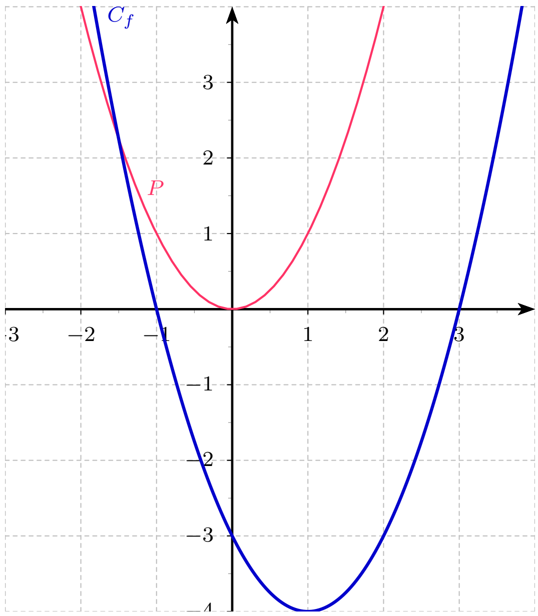
1. Study $f(x) = e^{\operatorname{sh}(x)} - (1+x)$ on $[-1;1]$ (It could interesting to study f").
   Let's deduce $\forall x \in]0;1[\,, 1 + x \leqslant e^{\operatorname{sh} x} \leqslant \dfrac{1}{1-x}$.
2. Let $k \in \mathbb{N},\ k \geqslant 2$. For all $n \in \mathbb{N}$, we put $u_n = \displaystyle\sum_{p=n}^{kn} \operatorname{sh}\left(\frac{1}{p}\right)$, evaluate $\displaystyle\lim_{n\to+\infty} u_n$.

Exercise 9.

Let's solve :

$$\begin{cases} \operatorname{ch} x + \operatorname{ch} y = \dfrac{35}{12} \\ \operatorname{sh} x + \operatorname{sh} y = \dfrac{25}{12} \end{cases}$$

Exercise 10.
Let's define f a real-valued function by $f(x) = x^2 - 2x - 3$. You will find below its graph in a Cartesian coordinate plane, and the graph of $y = x^2$:



- Find the geometrical transformation used to get C_f from P ?
- Prove it.

Exercise 11.

1. Sketch the graph of f defined by

$$f(x) = \begin{cases} 2x+5 & \text{si } x \in [-3,-2] \\ 1 & \text{si } x \in [-2,1] \\ -x+2 & \text{si } x \in [1,5] \end{cases}$$

2. Let's define h by $h(x) = -f(x)$.
   Find the geometrical transformation used, sketch the graph of h and give its expression.
3. Same question for $i(x) = f(-x)$
4. Same question for $g(x) = f(x+1) + 2$

Exercise 12.

1. Sketch as quick as possible those graphs (we will start with the graph of the elementary function used (sine, cosine, *etc.*).
   $f_1(x) = \sin(x) + 1$; $f_2(x) = -\cos(x)$; $f_3(x) = \ln(-x)$; $f_4(x) = 2\sqrt{x}$; $f_5(x) = \sin(2x)$; $f_6(x) = \sqrt{x+1}$
2. More difficult : $f_7(x) = 2\sin(x) + 1$; $f_8(x) = \ln(2x+1)$; $f_9(x) = \sin(2x) + 1$; $f_{10}(x) = 2\ln(x+1)$.

Exercise 13.
Find the smallest period of those functions :

1. $f_1(x) = \sin(3x)$
2. $f_1(x) = \cos(\omega x + \frac{\pi}{4})$
3. $f_1(x) = \sin(3x) - \cos(\frac{2x}{3})$
4. $f_1(x) = \frac{\tan(4x)}{\tan(2x)}$

Exercise 14.
Solve

1. $\sin(2x) = \sin(\frac{\pi}{3})$
2. $\cos(3x + \pi) = \cos(\frac{\pi}{2})$
3. $\tan(3x) = 1$
4. $\sin x + \sin(2x) = 0$

Exercise 15.
Solve on $\mathbb{R}$.

1. $\sin(2x) \leqslant \frac{1}{2}$
2. $\cos(3x + \pi) > -\frac{\sqrt{2}}{2}$
3. $\tan(3x) > 1$