Study of usual functions

Objectives

- **Know and apply the procedure for analyzing a function.**
- **Be familiar with standard functions.**

1 Conducting a function analysis

1. Determine **the domain** of f.
2. Determine **the interval of analysis** for f : depending on the properties of f, it may be possible to analyze f over an interval smaller than the domain ; this is then called the interval of analysis.

   - Is f even or odd ?

     Definition 1. Even function
     Let f be a function defined on a set $I \subseteq \mathbb{R}$. We say that f is an even function if :
     - I is symmetric about the origin.
     - $\forall x \in I, \ f(-x) = f(x)$

     Example 1.
     Are the following functions even on their domains ?
     $f(x) = \dfrac{x^2 + \cos x + 1}{-|x| - 1}$; $g(x) = 2x + 2$

     Property 1.
     A function is even if and only if its graph in an orthogonal coordinate system is symmetric about the y-axis.

     Example 2.
     Verify the previous property using a graph.

     Definition 2. Odd function
     Let f be a function defined on a set $I \subseteq \mathbb{R}$. We say that f is an odd function if :
     - I is symmetric about the origin.
     - $\forall x \in I, \ f(-x) = -f(x)$

     Example 3.
     Are the following functions odd on their domain of definition ?
     $f(x) = \dfrac{-x^3 + \sin x}{x^2 + 1}$; $g(x) = 2x + 2$

Property 2.
A function is odd if and only if its graph in an orthogonal coordinate system is symmetric with respect to the origin.

Example 4.
Verify the previous property using a graph.

- Is f periodic ? We will study this in the section on trigonometric functions.

3. Determine **the limits** at the boundaries of the domain of definition.
4. Determine **the domain of differentiability**, calculate the **derivative**, and draw up **the table of variations**. Specify key values : the boundaries of the domain of study, and the points where the derivative is zero or undefined.
5. Notable graphical features

- **Infinite branches :**
An infinite branch exists if x or $f(x)$ tends towards $+\infty$ or $-\infty$.
  - Case 1 : $\lim\limits_{x \to x_0} f(x) = \pm\infty$: vertical asymptote $x = x_0$
  - Case 2 : $\lim\limits_{x \to \pm\infty} f(x) = y_0$: horizontal asymptote $y = y_0$
  - Case 3 : $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$. Further details on this aspect can be found in the chapter on asymptotic expansions. In certain simple cases, you can determine an equivalent for the function as $x \to \pm\infty$. The shape of the curve resembles that of the equivalent found.

Example 5.
Determine the infinite branches of the function f defined by $f(x) = \dfrac{2-3x^2}{x+3}$

- **Notable tangents :**
  (a) If the derivative is zero : horizontal tangent
  (b) If the derivative or the rate of change tends to $+\infty$ or $-\infty$: vertical tangent ; point of non-differentiability.
  (c) One-sided tangents : obtained by considering f only to the right or only to the left at a point x_0

1.1 Image of an interval under a continuous function

Definition 3.
Let I be a subset of $\mathbb{R}$. The image of I under f is the set of all real numbers $f(x)$ where $x \in I$. This set is denoted by $f(I)$, so $f(I) = \{f(x) \mid x \in I\}$.

Example 6.
Determine $f(I)$ with $I = [-1; 3]$ eand f the square function.

Property 3.
The image of an interval under a continuous function is an interval.

Example 7.
Find the image of $\left[\dfrac{\pi}{4}; \dfrac{7\pi}{4}\right]$ under the sine function.

Property 4.
Let f be a continuous function on an interval $[a;b]$.
If f is increasing, the image of $[a,b]$ under f is $[f(a);f(b)]$.
If f is decreasing, the image of $]a,b]$ under f is $[f(b);f(a)[$.

Example 8.
Show that if f is continuous and not monotonic, the previous property does not hold.

Remarque 1.

- If f is defined on $]a;b]$ and has a limit at a, then $f(a)$ is replaced by $\lim_{x \longrightarrow a} f(x)$.
- The previous property can also be applied to any type of interval.

Remarque 2. The previous proposition allows one to determine the image of I under f using a variation table.

Example 9. Re-derive the results of the previous examples using a variation table.

1.2 Functions bounded above, bounded below, and bounded

Let f be a real-valued function defined on a domain denoted by D.

Definition 4. *Upper bound, least upper bound (supremum), and maximum.*
Let f be a bounded function on an interval $[a,b]$.

- **An** upper bound of f on $[a,b]$ is a real number M such that for all $x \in [a,b]$, $f(x) \leqslant M$.
- **The** least upper bound (or supremum) of f on $[a,b]$ is the smallest of the upper bounds. It is denoted by $\underset{x\in[a;b]}{sup} f(x)$.
- The maximum of f on $[a,b]$ is a real number M such that for all $x \in [a,b]$, $f(x) \leqslant M$, and there exists $x_0 \in [a,b]$ such that $M = f(x_0)$. It is denoted by $\underset{x\in[a;b]}{max} f(x)$.

The concepts of lower bound, greatest lower bound (infimum), and minimum are defined in the same way.

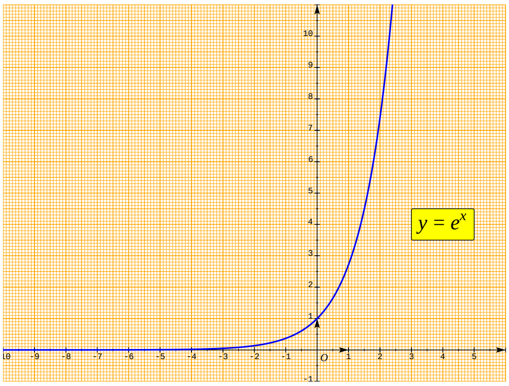
Example 10.
Are the following statements true or false ?

1. An upper bound is a maximum.
2. A maximum is an upper bound.
3. A bounded function always has a maximum on $[a,b]$.
4. A bounded function always has a least upper bound on $[a,b]$.

Example 11.
Prove that the function f defined on $\mathbb{R}$ by $f(x) = \dfrac{|\sin(x)|}{2+\cos(x)}$ is bounded on $\mathbb{R}$.

2 Exponentials, logarithms, powers

2.1 Exponential



Definition 5.
There exists a unique function from $\mathbb{R}$ to $\mathbb{R}$, called the exponential function and denoted by exp, which is differentiable on $\mathbb{R}$ such that :

$$\begin{cases} \exp(0) = 1 \\ \exp'(x) = \exp(x), \ \forall x \in \mathbb{R} \end{cases}$$

By definition, exp is continuous and differentiable on $\mathbb{R}$.

Functional equation

$$\forall x, y \in \mathbb{R} \ : \begin{cases} \exp(x+y) = \exp(x). \exp(y) \\ \exp(-x) = \dfrac{1}{\exp(x)} \end{cases}$$

We define $e = \exp(1)$ and denote $\exp(x)$ as e^x.

Variation
$\forall x \in \mathbb{R}, \exp(x) > 0$; since $\exp' = \exp$, the function exp is strictly increasing.

Limits at the boundaries

$$\lim_{x \to +\infty} e^x = +\infty \qquad \lim_{x \to -\infty} e^x = 0$$

The curve of exp has a horizontal asymptote at $-\infty$ with the equation $y = 0$, i.e., the x-axis.

Comparative growth rates

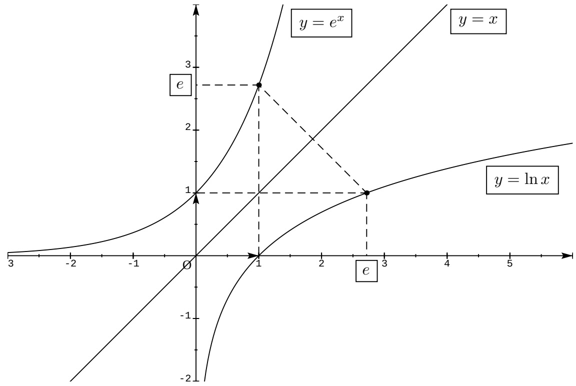
$$\begin{cases} \forall \alpha \in \mathbb{R}^{+*}, \lim\limits_{x \to +\infty} \dfrac{e^x}{x^\alpha} = +\infty \\ \forall \alpha \in \mathbb{R}^{+*}, \lim\limits_{x \to -\infty} |x|^\alpha e^x = 0 \end{cases}$$

Limit to know

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

2.2 Natural logarithm

Definition 6.
The natural logarithm function, denoted by ln, is the inverse function of exp, since exp is a bijection from $\mathbb{R}$ to $]0;+\infty[$. The function ln is therefore defined from $]0;+\infty[$ to $\mathbb{R}$.



Variation
ln follows the same pattern of variation as exp ; thus, ln is continuous, differentiable, and strictly increasing on $]0;+\infty[$.

Derivative

$$\forall x>0, \ln'(x) = \frac{1}{\exp'(\ln x)} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Limits at the boundaries

$$\lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to+\infty} \ln x = +\infty$$

The curve of ln has a vertical asymptote at 0.

Functional equation

$$\forall x,y>0 : \begin{cases} \ln(xy) = \ln(x)+\ln(y) \\ \ln\dfrac{1}{x} = -\ln x \end{cases}$$

Comparative growth rates

$$\begin{cases} \forall \alpha \in \mathbb{R}^{+*}, \lim\limits_{x\to+\infty} \dfrac{\ln x}{x^\alpha} = 0 \\ \forall \alpha \in \mathbb{R}^{+*}, \lim\limits_{x\to 0^+} x^\alpha \ln x = 0 \end{cases}$$

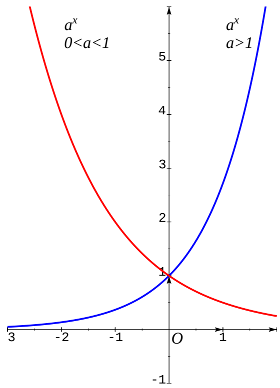
Limit to know

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

Example 12. Study the function $f : x \mapsto \ln(1+e^x)$

2.3 Exponential and logarithmic functions with arbitrary base

2.3.1 Exponential functions with arbitrary base

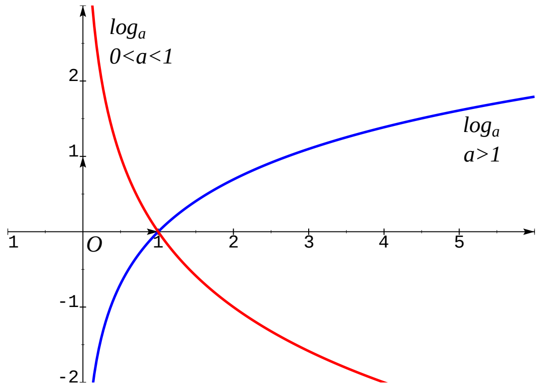


Definition 7.
Let $a > 0$. For any $x \in \mathbb{R}$, the exponential function with base a is defined as :

$$a^x = \exp(x \ln a) = e^{x \ln a}$$

Thus, studying an exponential function with base a reduces to studying a standard exponential function of the form $e^{\alpha x}$.

2.3.2 Logarithm functions with arbitrary base



Definition 8.
Let $a > 0$ and $a \neq 1$. For any $x > 0$, the logarithm with base a is defined as :

$$\log_a(x) = \frac{\ln x}{\ln a}$$

Similarly, studying a logarithm function with base a reduces—up to a multiplicative factor—to studying the ln function.

2.3.3 Decimal logarithm function

A logarithm function with base 10 is called the decimal logarithm and is denoted by log. This function is the inverse of the exponential function with base 10 : $x \to 10^x$. It is therefore used when dealing with powers of 10.

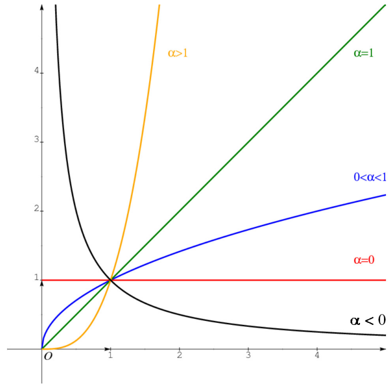
Example 13. For example, in chemistry, we have the formula : $[H^+] = 10^{-\mathrm{pH}}$. Deduce the pH in terms of the $[H^+]$ concentration.

2.4 Power functions

Definition 9.
For $\alpha \in \mathbb{R}$, we define :

$$f_\alpha : \left(\begin{array}{c} \mathbb{R}_+^* \to \mathbb{R} \\ x \mapsto x^\alpha \end{array} \right)$$

with $x^\alpha = e^{\alpha \ln x}$.
Thus, here too, the problem reduces to the study of a standard exponential function, except in cases where α is a natural number (standard power function), a negative integer (reciprocal of a standard power function), or a rational number, where we have : $x^{\frac{p}{q}} = \sqrt[q]{x^p}$.



2.5 Comparative growth

Let $\alpha > 0$ and $a, b > 1$: we have :

$$\lim_{x\to+\infty} \frac{a^x}{x^\alpha} = +\infty \qquad \lim_{x\to+\infty} \frac{\log_b x}{x^\alpha} = 0 \qquad \lim_{x\to 0} x^\alpha \log_b x = 0$$

This summarize as follows : $+\infty$:

$$\log_b x \ll x^\alpha \ll a^x$$

3 Circular functions

3.1 Periodic function

Definition 10.
Let T be a real number, and f a function defined on a set $I \subseteq \mathbb{R}$. A function f is said to be T-periodic, or to have period T, if :

- $\forall x \in I,\ x + T \in I.$
- $\forall x \in I,\ f(x + T) = f(x)$

Example 14.
Let f be the function defined by $f(t) = \cos(\omega t)$ for real t. Show that f has period $\dfrac{2\pi}{\omega}$. In physics, ω is called the angular frequency.

Proposition 1.
Let a, b, and ω be three real numbers. Then there exist three real numbers ϕ, ϕ', and A such that :

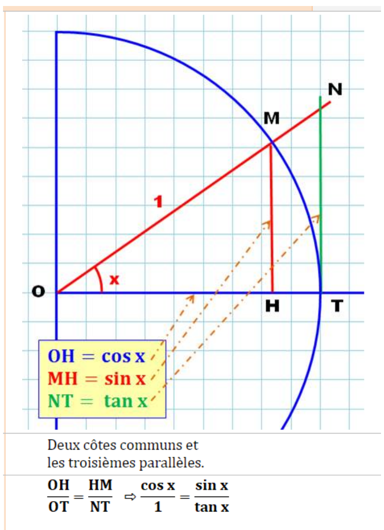
$$\text{for any real number } t : a\cos(\omega t) + b\sin(\omega t) = A\sin(\omega t + \phi) = A\cos(\omega t + \phi')$$

We have : $A = \sqrt{a^2 + b^2}$, $\tan\phi = \dfrac{a}{b}$, and $\tan\phi' = -\dfrac{b}{a}$

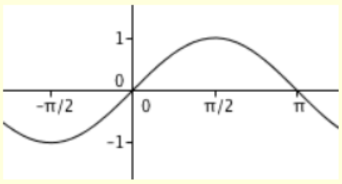
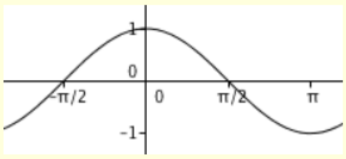
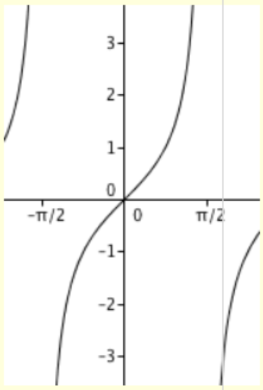
The previous property translates to the following in physics : the sum of two sinusoidal signals with the same angular frequency (and thus the same period) is a sinusoidal signal with the same angular frequency (and thus the same period) and a phase shift of ϕ.

Example 15.
Prove the previous property, then write the expression $\cos 2x + \sin 2x$ in the form $A\sin(\omega x + \phi)$.

3.2 The unit circle



3.3 Common trigonometric functions

Nom	sinus	cosinus	tangente
Notation	$\sin x$	$\cos x$	$\tan x$
Départ et arrivée	$\mathbb{R} \to [-1, 1]$	$\mathbb{R} \to [-1, 1]$	$\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\} \to \mathbb{R}$
Parité	Impaire	Paire	Impaire
Période	2π	2π	π
Dérivée	$\cos x$	$-\sin x$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$
Monotonie	Croissante sur $[-\pi/2, \pi/2]$	Décroissante sur $[0, \pi]$	Croissante sur $]-\pi/2, \pi/2[$
Courbe représentative			

Example 16.
Study the function $f : x \mapsto \tan x - \dfrac{1}{\tan x}$

3.3.1 Notable values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	

3.3.2 Trigonometry formulas

$\cos(-x) = \cos(x)$	$\sin(-x) = -\sin(x)$
$\cos(\pi - x) = -\cos(x)$	$\sin(\pi - x) = \sin(x)$
$\cos(\pi + x) = -\cos(x)$	$\sin(\pi + x) = -\sin(x)$
$\cos(\frac{\pi}{2} - x) = \sin(x)$	$\sin(\frac{\pi}{2} - x) = \cos(x)$
$\cos(\frac{\pi}{2} + x) = -\sin(x)$	$\sin(\frac{\pi}{2} + x) = \cos(x)$

3.3.3 Sum

- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\sin(a+b) = \cos(a)\sin(b) + \cos(b)\sin(a)$

3.3.4 Linéarisation

- $\cos^2(a) = \dfrac{1+\cos(2a)}{2}$
- $\sin^2(a) = \dfrac{1-\cos(2a)}{2}$
- $\sin(a)\cos(a) = \dfrac{1}{2}\sin(2a)$

3.3.5 solving a trigonometry equation

Solving an equation as $\cos(x) = \cos(a)$

$$\cos(a) = \cos(b) \Leftrightarrow \begin{cases} a = b + 2k\pi, \quad k \in \mathbb{Z} \quad (1) \\ \text{ou} \\ a = -b + 2k'\pi, \quad k' \in \mathbb{Z} \quad (2) \end{cases}$$

Solving an equation as $\sin(x) = \sin(a)$

$$\sin(a) = \sin(b) \Leftrightarrow \begin{cases} a = b + 2k\pi, \quad k \in \mathbb{Z} \quad (1) \\ \text{ou} \\ a = \pi - b + 2k'\pi, \quad k' \in \mathbb{Z} \quad (2) \end{cases}$$

Solving an equation as $\tan(x) = \tan(a)$

$$\tan(a) = \tan(b) \Leftrightarrow a = b + k\pi, \ k \in \mathbb{Z}$$

4 Hyperbolic functions

4.1 Hyperbolic cosine and sine

Definition 11. By analogy with Euler's formulas, the hyperbolic cosine and hyperbolic sine are the functions from $\mathbb{R}$ to $\mathbb{R}$ defined by :

$$\begin{cases} \cosh : x \mapsto \dfrac{e^x + e^{-x}}{2} \\ \sinh : x \mapsto \dfrac{e^x - e^{-x}}{2} \end{cases}$$

Continuity and differentiability
cosh and sinh are continuous and differentiable on $\mathbb{R}$.

Complete the properties of cosh and sinh.

Parity

- The function cosh is an .... function.
- The function sinh is an .... function.

Derivative

$$\forall x \in \mathbb{R}, \begin{cases} \cosh'(x) = \\ \sinh'(x) = \end{cases}$$

Variation table

x	
$\mathrm{ch}\, x$	

x	
$\mathrm{sh}\, x$	

Sign
For all $x \in \mathbb{R}$, $\cosh(x)$ is ....
$\sinh(x)$ changes sign at ....

Hyperbolic trigonometry
There are many formulas relating cosh and sinh, but this one is **fundamental** :

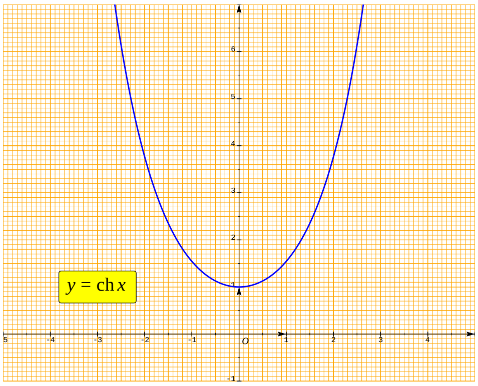
$$\forall x \in \mathbb{R}, \cosh^2 x - \sinh^2 x = 1$$

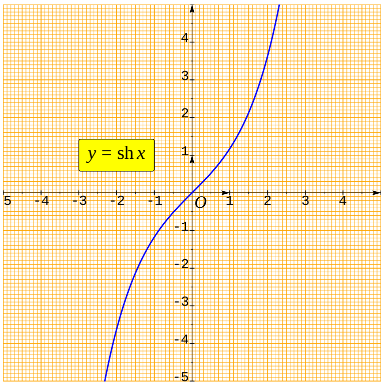
Example 17. Prove the above equality.

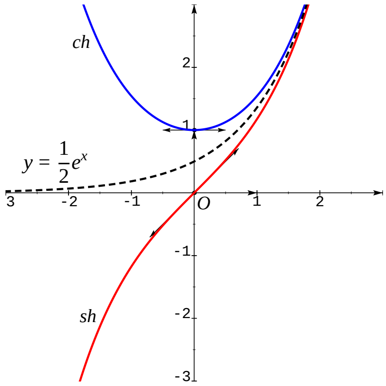
Comparison
Since $\cosh x - \sinh x = e^{-x}$ for all $x \in \mathbb{R}$, we have : $\lim_{x \to +\infty} (\cosh x - \sinh x) = 0$. The curves of the functions cosh, sinh, and $x \mapsto \dfrac{1}{2} e^x$ are said to be asymptotic.

Proposition 2.
$\forall x \in \mathbb{R}, \cosh(2x) = \cosh^2 x + \sinh^2 x$ and $\sinh(2x) = 2\cosh x \sinh x$

Example 18. Prove the above property.







4.2 Hyperbolic tangent

Definition 12. We call the hyperbolic tangent the function th defined on $\mathbb{R}$ by :

$$\forall x \in \mathbb{R}, \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

Continuity, differentiability
The function th is continuous and differentiable on $\mathbb{R}$.

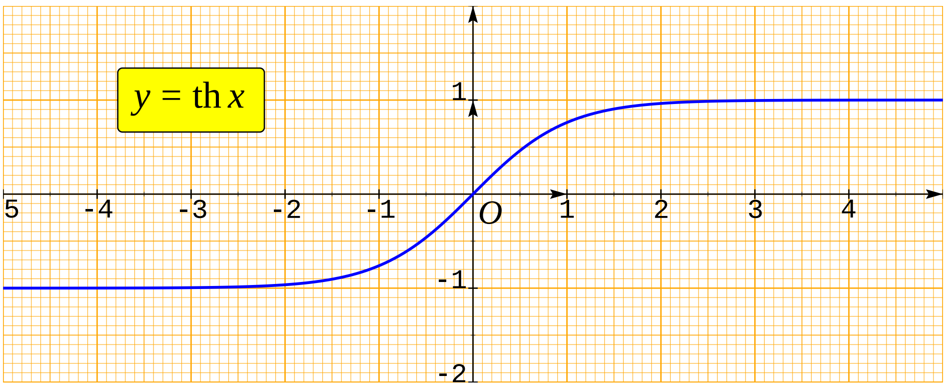
Parity
The function th is ....

Derivative

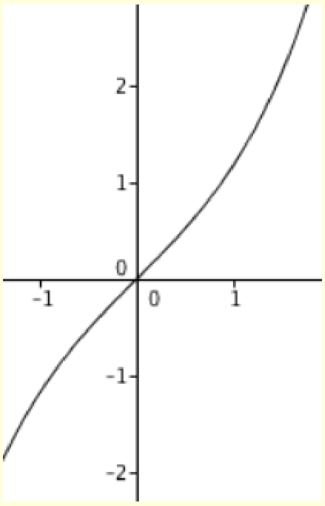
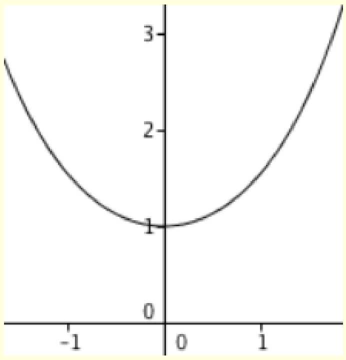
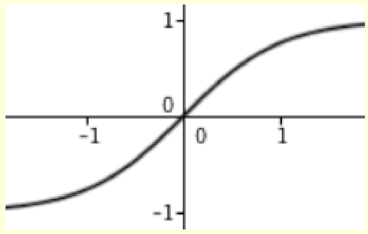
$$\forall x \in \mathbb{R}, \operatorname{th}'(x) = \text{.......................}$$

Variation table

x	
th x	



4.2.1 Summary of hyperbolic functions

Nom	sinus hyperbolique	cosinus hyperbolique	tangente hyperbolique
Définition	$\text{sh}x = \frac{e^x - e^{-x}}{2}$	$\text{ch}x = \frac{e^x + e^{-x}}{2}$	$\text{th}x = \frac{\text{sh}\,x}{\text{ch}\,x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
Départ et arrivée	$\mathbb{R} \to \mathbb{R}$	$\mathbb{R} \to [1, +\infty[$	$\mathbb{R} \to]-1, 1[$
Parité	Impaire	Paire	Impaire
Dérivée	$\text{ch}x$	$\text{sh}x$	$1 - \text{th}^2 x = \frac{1}{\text{ch}^2\,x}$
Monotonie	Croissante	Croissante sur $\mathbb{R}_+$	Croissante
Limites	$\lim_{x \to +\infty} \text{sh}x = +\infty$	$\lim_{x \to +\infty} \text{ch}x = +\infty$	$\lim_{x \to +\infty} \text{th}x = 1$
Courbe représentative			
Formules	$\text{ch}^2(x) - \text{sh}^2(x) = 1$		

5 Graph Transformations

5.1 Introduction

Part A) Experimentation

1) Sketch the graph of the exponential function.
2) Sketch the graphs of the following four functions on paper : $-\exp(x)$; $\exp(x)+1$; $\exp(x)-1$; $2\exp(x)$.
3) What geometric transformations map the graph of the exponential function to the sketched graphs ?
4) Answer the same questions for the four functions : $\exp(-x)$; $\exp(x+1)$; $\exp(x-1)$; $\exp(2x)$.

Part B) Śtatement of correspondences

Match each formula to its corresponding geometric transformation, specifying the details if possible.

(a) $-f(x)$;	(1) vertical shift up ;
(b) $f(x)+1$;	(2) vertical shift down ;
(c) $f(x)-1$;	(3) horizontal shift left ;
(d) $2f(x)$;	(4) horizontal shift right ;
(e) $f(-x)$;	(5) reflection across the x-axis ;
(f) $f(x+1)$;	(6) reflection across the y-axis ;
(g) $f(x-1)$;	(7) vertical stretch by a factor of 2 ;
(h) $f(2x)$.	(8) horizontal compression by a factor of 1/2.

Part C)

1) Which formula corresponds to a homothety (scaling transformation) with a ratio of 2, centered at the origin ?
2) To a rotation of half a turn, centered at the origin ?

5.2 Summary

Some functions have analytical expressions constructed from a standard function. These functions are said to be associated with a reference function. Studying this type of functional relationship allows their graphs to be obtained quickly and easily.
Let f and g be two numerical functions, defined on D_f and D_g respectively, with graphs $\mathcal{C}_f$ and $\mathcal{C}_g$ respectively in the orthonormal coordinate system $(O, \vec{i}, \vec{j})$.

5.2.1 Translations

Theorem 1 (Vertical translation)**.**
If $g(x) = f(x) + k$ with $k \in \mathbb{R}$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under the vertical translation by vector $k\vec{j}$.

Theorem 2 (Horizontal translation)**.**
If $g(x) = f(x+k)$ with $k \in \mathbb{R}$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under the horizontal translation by vector $-k\vec{i}$.

5.2.2 Symmetries

Theorem 3 (Symmetry about the x-axis)**.**
If $g(x) = -f(x)$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under reflection across the x-axis.

Theorem 4 (Reflection across the y-axis)**.**
If $g(x) = f(-x)$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under reflection across the y-axis.

Theorem 5 (Point reflection centered at O)**.**
If $g(x) = -f(-x)$
Then $\mathcal{C}_g$ is the image of $\mathcal{C}_f$ under point reflection centered at O.

5.2.3 Scaling

Theorem 6 (Vertical scaling)**.**
If $g(x) = k \cdot f(x)$ with $k > 0$
Then C_g is the image of C_f under vertical scaling by a factor of k.

Theorem 7 (Horizontal scaling)**.**
If $g(x) = f(k \cdot x)$ with $k > 0$
Then C_g is the image of C_f under horizontal scaling by a factor of $\dfrac{1}{k}$.

Theorem 8 (Homothety centered at O with ratio k)**.**
If $g(x) = k \cdot f\left(\dfrac{1}{k} \cdot x\right)$ with $k > 0$
Then C_g is the image of C_f under a homothety centered at O with ratio k.

Exercises

Tutorial 1-2

Exercise 1.
Analyze the following functions :

1. $f(x) = \sqrt{x^2 + x + 1}$
2. $f(x) = xe^{\frac{1}{x}}$
3. $f : x \mapsto x^{\frac{1}{x}}$

Exercise 2.
Determine whether the following functions are bounded above, bounded below, and/or bounded on an interval I of your choice :

1. $f(x) = x^2 e^{-x}$
2. $f(x) = \dfrac{1}{3x^2 - 2x + 1}$
3. $f(x) = \dfrac{x^2 \cos(x)}{x^2 + 1}$

Exercise 3.

1. Determine the image of $]-5; 0]$ under the function f defined by $f(x) = \dfrac{1}{x-1}$.
2. Solve the inequality $-3 \leqslant x^2 < 7$ in $\mathbb{R}$.
3. Determine the set of real numbers I such that $f(I) = J$ for the function f defined by $f(x) = e^{-x^2}$ and $J = [-2; 5]$.

Exercise 4. Let $n \geqslant 1$ be an integer. For each of the following functions, determine the supremum, the infimum, and the minimum/maximum (if they exist) on the given interval.

1. $f_n(x) = x^n e^{-x}$ on $[0, +\infty[$.
2. $g_n(x) = \dfrac{e^{xn} - 1}{x}$ on $]0, +\infty[$.

6 TD3-4

Exercise 5.
Calculate the exact values of these expressions
$\cos(\frac{-3\pi}{4})$ $\cos(\frac{5\pi}{6})$ $\sin(\frac{123\pi}{6})$ $\cos(\frac{\pi}{12})$ $\sin(\frac{\pi}{12})$

Exercise 6.
Find the smallest period of the following functions :

1. $f_1(x) = \sin(3x)$
2. $f_1(x) = \cos(\omega x + \frac{\pi}{4})$
3. $f_1(x) = \sin(3x) - \cos(\frac{2x}{3})$
4. $f_1(x) = \dfrac{\tan(4x)}{\tan(2x)}$

Exercise 7.
Solve the following equations in $\mathbb{R}$ and in $[0;2\pi]$.

1. $\sin(2x) = \sin(\frac{\pi}{3})$
2. $\cos(3x+\pi) = \cos(\frac{\pi}{2})$
3. $\tan(3x) = 1$
4. $\sin x + \sin(2x) = 0$

Exercise 8.
Solve the following inequalities on $\mathbb{R}$.

1. $\sin(2x) \leqslant \frac{1}{2}$
2. $\cos(3x+\pi) > -\frac{\sqrt{2}}{2}$
3. $\tan(3x) > 1$

7 Tutorial 5

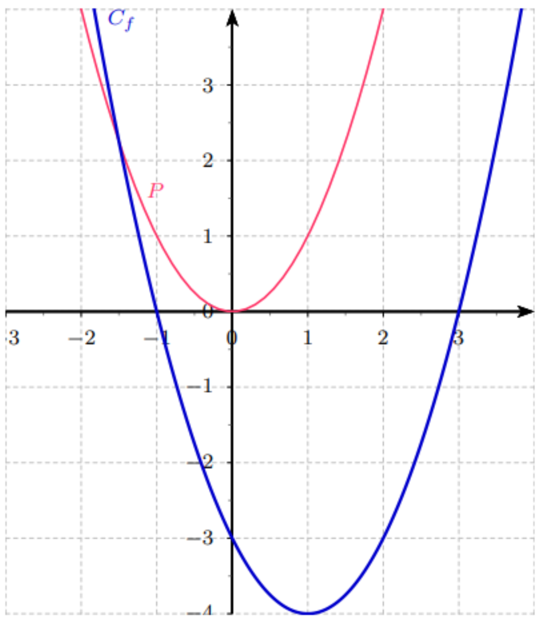
Exercise 9.
Study the following function :

1. $h : x \mapsto \cosh\left(\frac{2x-1}{x+1}\right)$

Exercise 10.
Express $\cosh(x)$ in terms of $\sinh(x)$ and $\sinh(2x)$.
Simplify $u_n = \prod_{p=1}^{n} \cosh\left(\frac{1}{2^p}\right)$, and deduce $\lim\limits_{n\to+\infty} u_n$.

8 Tutorial 6

Exercise 11.
Let f be defined on $\mathbb{R}$ by $f(x) = x^2 - 2x - 3$. Here is its graph in a Cartesian coordinate system, alongside the parabola with the equation $y = x^2$:



- What geometric transformation appears to yield C_f from P ?
- Prove this by calculation.

Exercise 12.

1. Sketch the graph of the function f defined by

$$f(x) = \begin{cases} 2x+5 & \text{if } x \in [-3,-2] \\ 1 & \text{if } x \in [-2,1] \\ -x+2 & \text{if } x \in [1,5] \end{cases}$$

2. Let h be the function defined by $h(x) = -f(x)$.
   Specify the associated transformation, then provide the graph and the expression for h.
3. Same question for $i(x) = f(-x)$
4. Same question for $g(x) = f(x+1) + 2$

Exercise 13.

1. Sketch the graphs of the following functions as quickly as possible. Start by sketching the graph of the basic function involved (sine, cosine, *etc.*).
   $f_1(x) = \sin(x) + 1$; $f_2(x) = -\cos(x)$; $f_3(x) = \ln(-x)$
2. More difficult : $f_7(x) = 2\sin(x) + 1$; $f_8(x) = \ln(2x+1)$