FUNCTIONS : LIMITS AND CONTINUITY

Learning Objectives
- **To know real numbers**
- **To become familiar with absolute values**
- **To know definitions of limits**
- **To be able to compute limits**
- **To know the notion of continuity**

We will study in this sections limits of real-valued functions of one real variable.

1 The set of real numbers

1.1 Subsets of $\mathbb{R}$

Example 1. What is a number ?

Definition 1.
We distinguish several subsets of real numbers.
- The set of natural numbers or positive integers, denoted by $\mathbb{N}$.
- The set of integers, denoted by $\mathbb{Z}$.
- The set of decimal numbers (its decimal expansion terminates after a finite number of digits),denoted by $\mathbb{D}$.
- The set of rational numbers (Quotient of two integers), denoted by $\mathbb{Q}$. The decimal expansion of a rational number either terminates after a finite number of digits or begins to repeat the same sequence of digits over and over.
- The set of irrational numbers (it means real numbers which are not rational).

Example 2.
Give an element for all previous sets and find inclusive relations between those sets.

It is important to distinguish all those sets as they have different applications.
For instance :

Set	In Mathematics	In Physics	Example
$\mathbb{N}$	Sequence Rank	Discrete Phenomena	
$\mathbb{D}$	Approximate value	Approximate Value	
$\mathbb{Q}$	Approximate value of an irrational number	Proportion	
$\mathbb{R}$	A real-valued function of one variable Sequence Values	Continuous Phenomena	

Question 1. Fill this array, giving a physical example in each case.

1.2 The absolute value

Definition 2.
Let x be a real number, its absolute value, denoted by $|x|$, is defined as follows :
$|x| = x$ if x is positive, $|x| = -x$ if not.

Example 3.
Express without the absolute value $\left|\frac{\sqrt{2}-1}{\sqrt{2}-3}\right|$.

Property 1.

- $|x| \geqslant 0$
- $|-x| = |x|$
- Triangular Inequality $||a| - |b|| \leqslant |a+b| \leqslant |a| + |b|$
- $\sqrt{x^2} = |x|$

Example 4.
Find examples where the above inequality is strict, and where the above inequality is an equality than demonstrate this inequality.

There is a link between absolute values and real intervals.

Property 2.
Let x, a and ε be real numbers.
- $|x-a| = \text{distance}(a; x)$
- $|x-a| \leqslant \varepsilon \Longleftrightarrow x \in [a-\varepsilon; a+\varepsilon]$

Example 5.
Draw on the real line the previous properties.

2 Introduction of the notion of limit on a physical example

2.1 Example : the force of gravitational interaction

In Newton's theory, , the force of gravitational interaction between two masses m_A and m_B is proportional to the product of the two masses and inversely proportional to the square of the distance between them d :

$$F_{AB} = G\frac{m_A m_B}{d^2}$$

and G is the gravitational constant.

2.2 Variables and functions

In our previous examples, there exists several letters. Those can be interpreted differently. We may use this formula to compute the force knowing others values.

Example 6.
Compute the force of gravitational interaction between two protons knowing that $m_A = m_B = 1,67\ 10^{-27}$kg, $G = 6,67\ 10^{-11}$ N.m^2.kg^{-2} et $d = 2,32\ 10^{-15}$ m.

Indeed, F depends on three inputs. In mathematics, we say that F is a **function** of three **variables** : m_A, m_B et d.

We may also fix two of those three inputs, for instance m_A et m_B. F will be a function of one variable d. m_A and m_B are called **constants**.

Example 7.
In mathematics, how do we denote the range of d under F ?

2.3 Domain of definition

We want to study F depending on d, thus we have to know what the set of real numbers such that $F(d)$ is defined. This set is called **domain of definition** of the function F. In our example, d can take any strictly positive value, thus the domain of definition of F is $]0;+\infty[$.

Remark 1.
Just looking at the definition $F(x) = G\dfrac{m_A m_B}{x^2}$, the domain of definition of F would be $\mathbb{R}^*$. The domain of definition of a function is different wheter we consider the physical or the mathematical context of the function.

Example 8.
Give the domain of definition of f defined by $f(x) = \dfrac{x^2+2x-3}{x^2-1}$

2.4 Limits of F when the variable approaches a real number

We can't compute $F(d)$ if $d = 0$, but we can compute $F(d)$ for d very close to 0. In mathematics, d can be as close as we want to 0.

Example 9.
What can you say about $F(d)$ when d approaches, as close as we want, 0 ? What does it mean in physics ?

2.5 Limit of F when the variable approaches infinity $+\infty$

We can compute $F(d)$ when d takes values as big as we want.

Example 10.
What can you say about $F(d)$ when d takes values as big as we want ? What does it mean in physics ?

3 Logic Quantifiers

3.1 Universal Quantifier $\forall$

"$\forall x \in E$" means :" **for all** element x in E".

Example 11.
Write using quantifiers : " the square of a real number is always positive".

3.2 Existential Quantifier $\exists$ and $\exists!$

- "$\exists x \in E$" means : " **it exists** at least an element x in E".
- " / " or " , " means : " such that".
- "$\exists!$", means there exists one and only one element ...(uniqueness quantifier), a unique element...

Example 12.
Write using mathematical quantifiers :
" The equation $2x^2 - x = 0$ has an integer solution."
" 1 has a unique fiber under the map ln."

3.3 Negating quantifiers

Theorem 1. The rules for negating quantifiers are

1. non $(\forall x \in E \quad \mathrm{P}(x)) \quad \Leftrightarrow \quad (\exists x \in E /$ non $\mathrm{P}(x))$
2. non $(\exists x \in E / \mathrm{P}(x)) \quad \Leftrightarrow \quad (\forall x \in E \quad$ non $\mathrm{P}(x))$

Example 13.
Express the negation of the following sentences :
P : $\exists x \in \mathbb{R} : f(x) = 3$.
Q : $\forall y \in \mathbb{R}$ $f(y)$ is an integer
R : $\forall y \in F \quad \exists x \in E \ / \ f(x) = y$

4 Definition of a limit

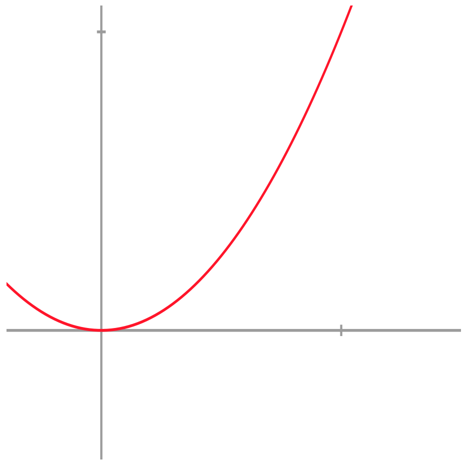
In this section, f is a real-valued function of one variable, defined on an interval I.
$a \in \mathbb{R} \cup \{-\infty, +\infty\}$

4.1 Finite limit at the point a

Definition 3.
Let $l \in \mathbb{R}$.
Notation : $\lim\limits_{x \to a} f(x) = l$ or $f(x) \underset{x \to a}{\to} l$, is read : "$f$ has the limit l at the point a".

1. First Case $a \in \mathbb{R}$: the limit of f as x approaches a is l if and only if :

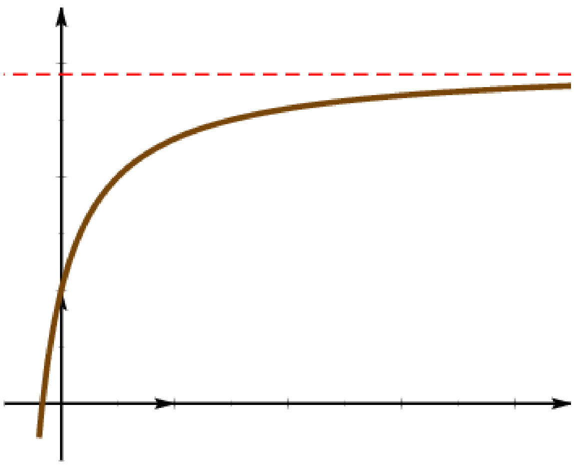
$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in I \backslash \{a\}, |x - a| \leqslant \alpha \Rightarrow |f(x) - l| \leqslant \varepsilon$$



Example 14. Write, using quantifiers, that $\lim_{x \to a} f(x) \neq l$.

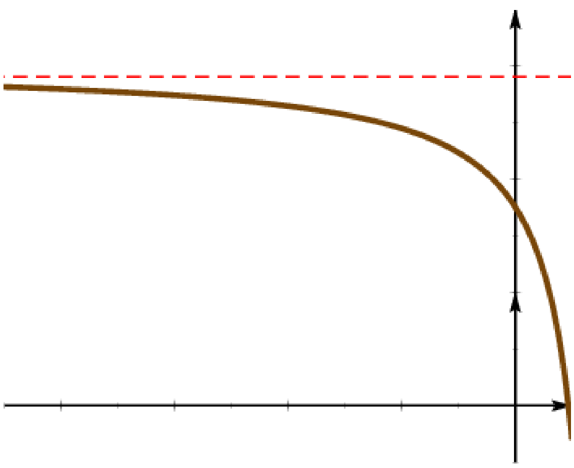
2. Second Case $a = +\infty$: the limit of f as x approaches $+\infty$ is l if and only if :

$$\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in I, x \geqslant A \Rightarrow |f(x) - l| \leqslant \varepsilon$$



3. Third Case $a = -\infty$: the limit of f as x approaches $-\infty$ is l if and only if :

$$\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in I, x \leqslant A \Rightarrow |f(x) - l| \leqslant \varepsilon$$



Intuitively, $\lim_{x \to a} f(x) = l$ means : whatever is $\varepsilon > 0$, to get $f(x)$ close enough to l with the term of error equal to $\leqslant \varepsilon$, it is sufficient to take x sufficiently close to a, in a neighborhood of a. Be careful as the neighborhood of a depends on ε.

Theorem 2. *Uniqueness of the limit*
Let's consider $f : I \to \mathbb{R}$ a function and l, l' two real numbers. Let's assume that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = l'$. Then $l = l'$.

Remark 2. The limit at the point a is unique but is not always equal to $f(a)$.

4.2 Infinite limit at the point a

In this section, a is a endpoint (finite or not) of the interval I.

Definition 4.
Let $f : I \to \mathbb{R}$ be a function.

1. First Case $a \in \mathbb{R}$: the limit of f as x approaches the point a is infinity $+\infty$ (respectively $-\infty$) if and only if :

$$\forall B \in \mathbb{R}, \exists \alpha > 0, \forall x \in I \backslash \{a\}, |x - a| \leqslant \alpha \Rightarrow f(x) \geqslant B \text{ (respectivement, f(x) } \leqslant \text{ B)}$$

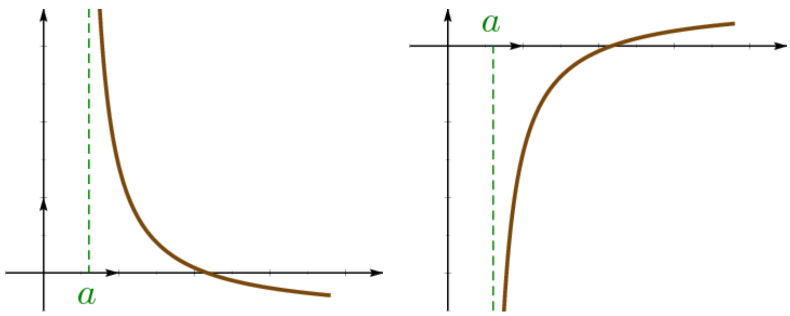
2. Second Case $a = +\infty$: the limit of f as x approaches $+\infty$ is is infinity $+\infty$ (respectively $-\infty$) if and only if :

$$\forall B \in \mathbb{R}, \exists A \in \mathbb{R}, \forall x \in I, x \geqslant A \Rightarrow f(x) \geqslant B \text{ (respectively, f(x) } \leqslant \text{ B)}$$

3. Third Case $a = -\infty$: the limit of f as x approaches $-\infty$ is $+\infty$ (respectively $-\infty$) if and only if :

$$\forall B \in \mathbb{R}, \exists A \in \mathbb{R}, \forall x \in I, x \leqslant A \Rightarrow f(x) \geqslant B \text{ (respectively, f(x) } \leqslant \text{ B)}$$

We denote it : $\lim_{x \to a} f(x) = +\infty$ respectively $\lim_{x \to a} f(x) = -\infty$



Example 15.
Let f be a function defined on $\mathbb{R}$ and a be a real number. Draw a function checking this property :

$$\exists \alpha \in \mathbb{R} : \forall \varepsilon > 0, |x - a| < \alpha \Longrightarrow |f(x) - l| \leqslant \varepsilon$$

4.2.1 Left and right hand limits

Definition 5. *Left hand limit*
Let's assume that x_0 is not the left endpoint of I. Studying the left hand limit of f at the point x_0 means that we study the values of f only for $x < x_0$ and $x \in I$. We denote it : $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x)$ or $\lim\limits_{x \to x_0^-} f(x)$

Example 16. Give $\lim\limits_{x \to 0^-} \dfrac{1}{x}$

Definition 6. *Right hand limit*
Let's assume that x_0 is not the right endpoint of I.Studying the right hand limit of f at the point x_0 means that we study the values of f only for $x > x_0$ and $x \in I$. We denote it : $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x)$ or $\lim\limits_{x \to x_0^+} f(x)$

Example 17. Give $\lim\limits_{x \to 0^+} \dfrac{1}{x}$

Property 3. We assume that x_0 is an element of I but is not one of its endpoints. f has a limit at the point x_0 if and only if the right hand limit is equal to the left hand limit. The limit of f at the point x_0 is the common value.

Remark 3.
If x_0 is one of the endpoints of I then :
- Either $I =]x_0; ...$ or $I = [x_0; ..$ then $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0^+} f(x)$.
- Either $I = ...; x_0[$ or $I = ...; x_0]$ then $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0^-} f(x)$.

Example 18. Let's define : $\delta : \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto \begin{cases} 1 \text{ si } x = 0 \\ \dfrac{|x|}{x} \text{ si } x \neq 0 \end{cases} \end{array}$

Does δ have a limit at 0 ?

5 Operations on limits

a is either a real number or infinite.

5.1 Limit of a sum

$\lim\limits_{x \to a} (f(x) + g(x)) =$

$\lim\limits_{x \to a} f(x) =$ / $\lim\limits_{x \to a} g(x) =$	$l \in \mathbb{R}$	$+\infty$	$-\infty$
$m \in \mathbb{R}$			
$+\infty$			
$-\infty$			

0. IF= Indeterminate Form, all cases are possible, we may not predict what happens

5.2 Limit of a product

$\lim_{x\to a} (f(x).g(x)) =$

$\lim_{x\to a} f(x) =$ / $\lim_{x\to a} g(x) =$	$l > 0$	$l < 0$	0	$+\infty$	$-\infty$
$m > 0$					
$m < 0$					
0					
$+\infty$					
$-\infty$					

5.3 Limit of a quotient

$\lim_{x\to a} \frac{f(x)}{g(x)} =$

$\lim_{x\to a} f(x) =$ / $\lim_{x\to a} g(x) =$	$l > 0$	$l < 0$	0	$+\infty$	$-\infty$
$m > 0$					
$m < 0$					
0^+					
0^-					
$\pm\infty$					

Example 19.
Give an example of an indeterminate form in each previous case.

5.4 Limits of composition of functions

Definition 7.
Let's assume that f is a unfction defined on a set I and g a function defined on $f(I)$. Then $g \circ f$ is the function defined on I by $g \circ f(x) = g(f(x))$ pour tout $x \in I$.

Example 20.

1. Give the expression of $f \circ g$ and $g \circ f$ with $f(x) = x + 3$ et $g(x) = x^2 - 1$.
2. Find two functions f and g such that $h = f \circ g$ with $h(x) = \sqrt{x^4 + 2x^2 + 1}$.

Theorem 3.
Let's consider $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ such that $f(I) \subset J$ thus the composition $g \circ f$ exists.

1. If f has a limit (finite or not) b at the point a then b is an element or a endpoint of J

2. Moreover if g has a limit (finite or not) l at the point b, then $g \circ f$ has the limit l at the point a :

$$\begin{cases} f(x) \underset{x \to a}{\longrightarrow} b \\ g(x) \underset{x \to b}{\longrightarrow} l \end{cases} \Rightarrow g\left(f(x)\right) \underset{x \to a}{\longrightarrow} l$$

Example 21.
Compute $\lim\limits_{x \to +\infty} \ln\left(1 + \dfrac{1}{x}\right)$

5.5 Limits at infinity for polynomial and rational functions

Theorem 4.
The limit at infinity $\pm\infty$ of a polynomial function P, which associates to each x a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, is equal to the limit at $\pm\infty$ of the term of higher degree of P : $a_n x^n$.
Thus we have :

$$\lim_{x \to \pm\infty} a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \lim_{x \to \pm\infty} a_n x^n$$

Example 22. Give the limit of $f(x) = 2x^2 + 3x - 1$ at $-\infty$.

Theorem 5.

There is a basic rule for evaluating limits at infinity for a rational function Q defined as the quotient of two polynomials : $Q(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$.
The limit at infinity $\pm\infty$ of Q is equal to the quotient of terms of higher degrees at the numerator over the terms of higher degrees at the denominator.
Thus we have :

$$\lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$$

Example 23.
Give the limit of $f(x) = \dfrac{2x^2 + 3x - 1}{5x^2 - x + 3}$ at $-\infty$.

5.6 Limits and square root functions

There exists two main examples :

- Indeterminate form $+\infty$ of type $\dfrac{\sqrt{x+3}}{\sqrt{x-4}}$: we factorize by $\sqrt{x}$ at the numerator and at the denominator.
- Indeterminate form of type $\sqrt{x+3} - \sqrt{2x+3}$: we multiply and divide by the "conjugate" $\sqrt{x+3} + \sqrt{2x+3}$.

Example 24.
Give the limits at $+\infty$ for the two previous examples.

6 Limits and Inequalities

6.1 Theorem

a is either a real number or infinite.

Theorem 6. *Limits and inequalities theorem*
Let f, g be two real valued functions defined on I. We assume that both f and g have finite limits l and m at some point a and that $f \leqslant g$ at the neighborhood of a then : $l \leqslant m$

Example 25.
If f, g are two real-valued functions defined I such that f and g have both finite limits l and m at the point a and such that $f < g$ at the neighborhood of a do we have : $l < m$?

Theorem 7. *The sandwich theorem (or squeeze theorem or pitching theorem*

Let f, g, h be three real valued functions defined on I. Let's assume that : $\begin{cases} f(x) \underset{x \to a}{\to} l \\ h(x) \underset{x \to a}{\to} l \end{cases}$ and $f \leqslant g \leqslant h$ at the neighborhood of a, then it follows $g(x) \underset{x \to a}{\to} l$

Example 26. Give the limit at $+\infty$ of $\dfrac{\cos x}{x}$

Property 4.

1. Let f, g be two real-valued functions defined on I. Let's assume that $f(x) \underset{x \to a}{\to} +\infty$ and $f \leqslant g$ at the neighborhood of a, then it follows $g(x) \underset{x \to a}{\to} +\infty$
2. Let f, g be two-real valued functions defined on I. Let's assume that $g(x) \underset{x \to a}{\to} -\infty$ and $f \leqslant g$ at the neighborhood of a, then $f(x) \underset{x \to a}{\to} -\infty$

Example 27.
Let's define $f : \mathbb{R} \to \mathbb{R}, x \mapsto x + \cos x$. What are : $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$?

7 Limit of a monotonic function

Theorem 8.
Let f be a function defined on an open interval $I =]a; b[$, with a and b in $\mathbb{R} \cup \{-\infty; +\infty\}$. Let's assume that f is a monotonic function on I, then f has a finite or infinite limit at the point a and at the point b.

Example 28.

1. What can you say about the limit of a function f in each case :
   (a) f is a strictly increasing function ?
   (b) f is also bounded above ?
2. Give necessary condition(s) on f so that f admits a finite limit ? Are those conditions sufficient ?

8 Continuity

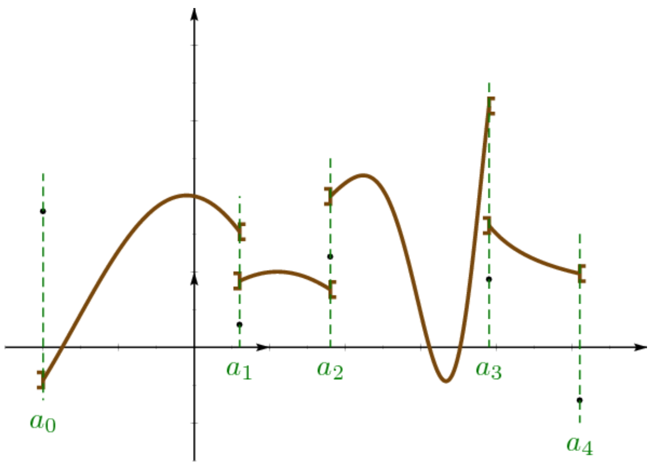
8.1 Continuous Function

We study the limit of a function at some point of its domain of definition

Definition 8.
Let $f : I \to \mathbb{R}$ be a function.

1. Let $x_0 \in I$. f is continuous at x_0 means that :
   - f has a finite limit at x_0
   - $\lim\limits_{x \to x_0} f(x) = f(x_0)$

   If one of those condition is not checked, f is said discontinuous at x_0.
2. f is continuous on I if and only if this function is continuous at all points x_0 of I.



Example 29.
Is the function δ previously defined 18 continuous at 0 ?

Property 5.
If f is continuous at a and if g is continuous at $f(a)$ then $g \circ f$ is continuous at a.

Example 30.
Let's define f by $f(x) = \ln(\sqrt{x} + 1)$. Is f continuous at 0 ?

Property 6.
Let's assume that f is a function defined as operations (sum, product...) of usual functions (sin called the sine, cos called the cosine, rational functions...). then f is continuous on its domain of definition.

Remark 4.
To prove that a function is continuous at some points we either use the previous proposition (useful for the points where f coincides with usual functions), or we use the 8.

Example 31.
Let f be the fonction defined on $\mathbb{R}$ by :

$$\begin{cases} f(0) = 2 \\ f(x) = \dfrac{\sin(x)}{x} \end{cases}$$

Is this function$f$ continuous on $\mathbb{R}$?

8.2 Continuous Extension

Definition 9.
Let's consider f a function defined on an open interval I, and let a be one of its endpoints. If f has a limit l at a, we **extend f by continuity at** a setting $f(a) = l$.

Example 32.
Let's define f on $]0; +\infty[$ by :

$$f(x) = \frac{\sin(x)}{x}$$

Is it possible to extend f at 0 ?

8.3 Image of an interval through a continuous function

Definition 10.
Let I be a subset of $\mathbb{R}$. The image of I under f is the set of real numbers $f(x)$ where $x \in I$. We denote it $f(I)$ so $f(I) = \{f(x)/x \in I\}$

Example 33.
Give $f(I)$ if $I = [-\sqrt{5}; 2,5]$ and f is the floor function.

Property 7.
The image of an interval under a continuous function is an interval.

Example 34.
Give the image of $[\frac{\pi}{4}; \frac{7\pi}{4}]$ under the sine.

Property 8.
The image of $[a, b]$ under a continuous and increasing function f is $[f(a); f(b)]$.
The image of $]a, b]$ under a continuous and decreasing function f is $[f(b); f(a)[$.

Remark 5.
The previous property can be applied for open, closed, left-opened, right-opened intervals.

Exercises Workout 1

Exercise 1.
Let A and B be two sets. We denote $A - B$ the set $\{x \in A \text{ and } x \notin B\}$.
Let's consider : $\mathbb{N}, \mathbb{Z} - \mathbb{N}, \mathbb{D} - \mathbb{Z}, \mathbb{Q} - \mathbb{D}$ and $\mathbb{R} - \mathbb{Q}$.
In each case, is it possible that c belongs to those previous sets ?

1. $c = a^b$ with a and b two positive real numbers.
2. $c = \sqrt{a}$ with a a positive real number.
3. $c = \dfrac{a + bd}{d^2 + a^2}$ with a, b and d three rational numbers, non equal to zero.

Exercise 2.
Right or Wrong ?

1. The quotient of two irrational numbers is always an irrational number.
2. In physics, numerical computations use decimals.
3. All real number has a scientific notation.

Exercise 3.
Let a and b two real positive numbers such that $a < b$.

1. Find a real number between $\sqrt{a}$ and $\sqrt{b}$.
2. Does $\sqrt{a} + \dfrac{\sqrt{b} - \sqrt{a}}{3}$ belong to $[\sqrt{a}; \sqrt{b}]$?
3. Can you give an infinite set of reals belonging to $[\sqrt{a}, \sqrt{b}]$?

Exercise 4.
Find the real numbers x such that :

1. $|x - 2| < 3$
2. $|x + 4| < -1$
3. $|x + 5| \leqslant 3$
4. $|x + 2| \geqslant 5$
5. $|x + 5| \geqslant -1$
6. $|x - 2| < 5$ and $|x - 3| \leqslant 4$
7. $|x + 1| < 1$ or $|x + 2| \leqslant 4$
8. $|x - 5| < 2$ and $|x + 1| \leqslant 1$
9. $|x - 2| < 4$ or $|x + 2| \leqslant 2$

Exercises Workout 2-3

Exercise 5.
Let a and b two real numbers, find the real numbers α and β (depending on a and b) such that $x \in [a; b] \iff |x - \alpha| \leqslant \beta$.

Exercise 6.
Find the points $M(x; y)$ such that :

1. $|x + y| = 2$
2. $|x + y| \leqslant 2$
3. $|x| + |y| \leqslant 3$

Exercise 7.
Give the domain of definitions of the following functions :

1. $f(x) = \dfrac{1}{x+2}$;
2. $f(x) = \dfrac{\sin x}{x}$
3. $f(x) = \sqrt{-2x+3}$

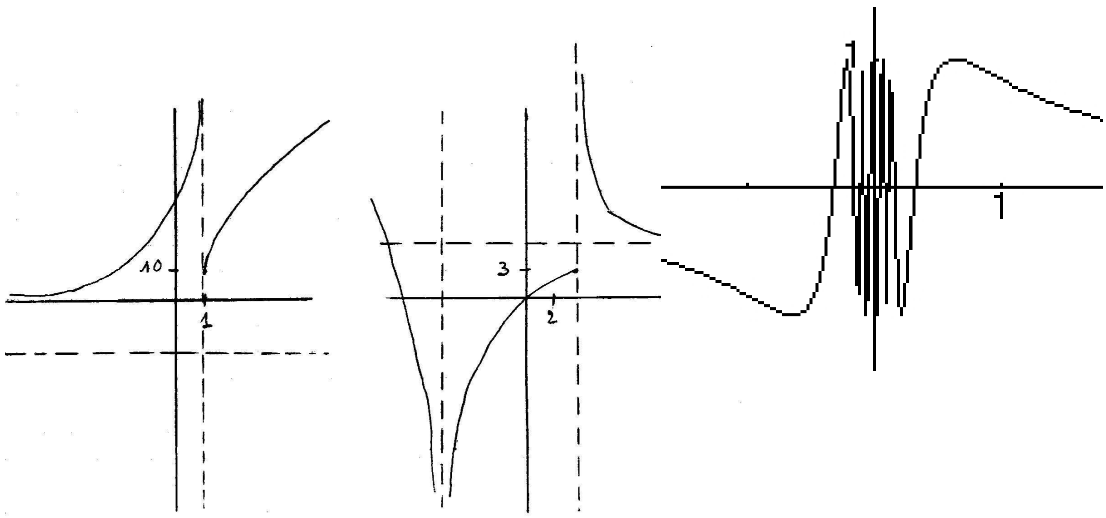
Exercise 8.
Give the mathematical notation for the functions below (f is the function and x the variable), then give the domain of definition.

1. $H = \dfrac{r+\omega}{\lambda\omega(1-\omega^2)}$ (H is called transfer function).
2. $z = 5t^2$ (z is the distance travelled by a ball in a freefall).
3. $P = \dfrac{nRT}{V}$ (P is the thermodynamic pressure).
4. $F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1q_2}{r^2}$ (F is the electrostatic field between charges).

Exercise 9.
In each case, draw the shape of the curve.

1. $\lim\limits_{2^+} f = 3$, $\lim\limits_{2^-} f = -\infty$, $\lim\limits_{+\infty} f = 5$ et $\lim\limits_{-\infty} f = +\infty$.
2. $\lim\limits_{-1^+} f = -\infty$, $\lim\limits_{-1^-} f = 0$, $\lim\limits_{-\infty} f = 0$ et $\lim\limits_{+\infty} f = -\infty$.

Exercise 10.
In the following examples, conjecture the limits of f ? Explain why this is just a conjecture.



Exercise 11.
If possible, link two columns.

	$\forall \varepsilon > 0, \exists \alpha > 0$ tel que $\forall x \in]5-\alpha; 5+\alpha[\quad \lvert f(x) - 5 \rvert \leqslant \varepsilon$
$\lim\limits_{x \to 5} f(x) = 5$	$\forall \varepsilon > 0, \forall \alpha > 0$ tel que $\forall x \in]3-\alpha; 3+\alpha[\quad \lvert f(x) - 5 \rvert \leqslant \varepsilon$
$\lim\limits_{x \to +\infty} f(x) = 5$	$\forall \varepsilon > 0, \exists \alpha > 0$ tel que $\forall x \in]5-\alpha; 5+\alpha[\quad \lvert f(x) - f(5) \rvert \leqslant \varepsilon$
$\lim\limits_{x \to 5} f(x) = +\infty$	$\exists \varepsilon > 0, \forall M \in \mathbb{R}$ tel que $\forall x > M \quad \lvert f(x) - f(5) \rvert \leqslant \varepsilon$
$\lim\limits_{x \to -\infty} f(x) = 5$	$\forall \varepsilon > 0, \exists M \in \mathbb{R}$ tel que $\forall x > M \quad \lvert f(x) - 5 \rvert \leqslant \varepsilon$
	$\exists \varepsilon > 0, \forall \alpha > 0$ tel que $\forall x \in]5-\alpha; 5+\alpha[\quad \lvert f(x) - f(5) \rvert \leqslant \varepsilon$

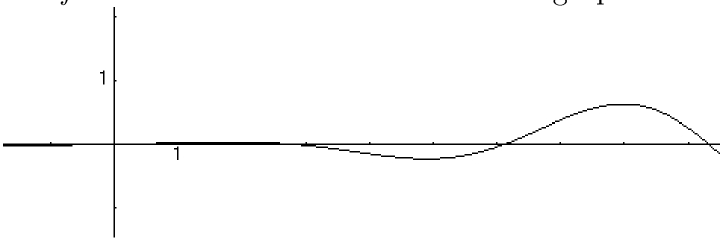
Exercise 12.
Write, using mathematical symbols, that a function has no finit limit at infinity.

Exercise 13.
Prove that if f has a finit limit in a then that limit is unique.

Exercise 14.
In each cas, draw a function checking the given property :

1. $\exists l \in \mathbb{R} : \forall (a,b) \in \mathbb{R}^2 \exists x \in [a,b] : f(x) = l$
2. $\forall (a,b) \in \mathbb{R}^2 \exists l \in \mathbb{R} \exists x \in [a,b] : f(x) = l$

Exercise 15.
Let f be the fonction associated to this graph :



1. Let's predict the limit of f in 0.
2. We have $f : f(x) = 10^{-2}(10 + x^2) \sin \dfrac{1+x^2}{x}$.
   Draw this function with you calculator. Do you get the same curve ?
3. Using the expression of f, draw the representation of f for $x \in [-1; 1]$.
4. Draw the previous curve on your calculator.
5. Is a graph reliable to find limits ?

Exercises Workout 4-5-6

Exercise 16. For each case, give the limit if it exists

$a)\ \lim\limits_{x \to 0} \dfrac{x^2 + 2\lvert x \rvert}{x}$ $\qquad b)\ \lim\limits_{x \to -\infty} \dfrac{x^2 + 2\lvert x \rvert}{x}$ $\qquad c)\ \lim\limits_{x \to 2} \dfrac{x^2 - 4}{x^2 - 3x + 2}$

$d)\ \lim\limits_{x \to \Pi} \dfrac{\sin^2 x}{1 + \cos x}$ $\qquad e)\ \lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1+x^2}}{x}$ $\qquad f)\ \lim\limits_{x \to +\infty} \sqrt{x+5} - \sqrt{x-3}$

$g)\ \lim\limits_{x \to 0} \dfrac{\sqrt[3]{1+x^2} - 1}{x^2}$ $\qquad h)\ \lim\limits_{x \to 1} \dfrac{x-1}{x^n - 1}$

Exercise 17. (optional)
Prove that the function cos has no finite limit in $+\infty$.

Exercise 18.
Find the limits of the following functions.

1. $\lim_{t\to+\infty} \frac{Rt^2+C}{R^2\omega t^2+C}$
2. $\lim_{t\to 0} \frac{Rt^2+C}{R^2\omega t^2+C}$
3. $\lim_{t\to+\infty} N_0 e^{-kt}$.

Exercise 19.
Give the limits of those functions in a, then give a physical interpretation of the result.

1. $H = \frac{r+\omega}{\lambda\omega(1-\omega^2)}$
   (a) ω is the variable and $a = +\infty$, $a = 1^+$.
   (b) r is the variable and $a = +\infty$.
2. $z = 5t^2$ $a = +\infty$
3. $P = \frac{nRT}{V}$, V is the variable and $a = +\infty$, and $a = 0^+$.
4. $F = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}$, $r$is the variable, $a = +\infty$, and $a = 0^+$.

Exercise 20.
Find the limits of the following functions.

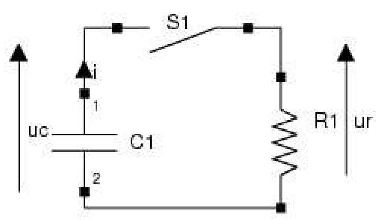
1. $\lim_{x\to+\infty} -2x - 3\cos x$
2. $\lim_{x\to-\infty} x + \sin x$
3. $\lim_{x\to+\infty} \frac{x+\cos x}{x-\sin x}$

Exercise 21.
Are the following functions continuous at a?

1. $f(x) = \frac{x}{\sqrt{|x|}}$ for $x \neq 0$ and $f(0) = 1$, $a = 0$.
2. $f(x) = \frac{x^2+x-2}{x^2+3x-4}$ for $x \neq 1$ and $x \neq -4$ and $f(1) = 10$, $a = 1$.

Exercise 22.
Find an example of a function f, such that f has a limit in a, but is not continuous at a.

Exercise 23.
We consider the electrical circuit below.



When this circuit is closed, we have $U_c = Ee^{-\frac{t}{\tau}}$ and $U_r = U_c$ with τ a positive constant.

1. Give the domain of definition of U_c and of U_r.
2. Study the continuity of those functions on the previous sets.
3. Find the limit of U_C when t tends to $+\infty$ and give a physical interpretation.

Exercise 24.
Is it possible to extend the following functions to the specified points ?

1. $f(x) = \dfrac{x^2-4}{x-2}$, $x = 2$
2. $f(x) = \dfrac{x^2-9}{\sqrt{x+3}}$, $x = -3$
3. $f(x) = \dfrac{\sqrt{3x^2+1}-2}{x-1}$, $x = 1$

Exercise 25.

1. Give the image of the set]-5 ;0] through the function f defined by $f(x) = \dfrac{1}{x-1}$.
2. Solve in $\mathbb{R}$ $-3 \leqslant x^2 < 7$.
3. Give the set of real numbers I such that $f(I) = J$, with the function f $f(x) = e^{-x^2}$ and $J = [-2; 5]$.

Exercise 26.
Let f be a fonction defined on $\mathbb{R}$.

1. Can we have both f continuous, strictly increasing f on the set $[a; b]$ and $f([a; b]) = [f(a); f(b)]$?
2. Is is possible to have a function f not continuous in $c \in]a; b[$, strictly increasing on the set $[a; b]$ and such that $f([a; b]) = [f(a); f(b)]$?
3. Give the reciprocal of the line 1 of property 8.
4. Is the reciprocal true ? ?

Exercise 27.
Let f be defined by $f(x) = (-1)^{E(x)}(x - E(x))$ where $E(x)$ denotes the floor function which associates to x the largest integer less than or equal to x. Study the continuity of f.