FUNCTIONS : LIMITS AND CONTINUITY

Learning Objectives
- **To know real numbers**
- **To become familiar with absolute values and inequalities**
- **To know definitions of limits**
- **To be able to compute limits**
- **To know the notion of continuity**

We will study in this sections limits of real-valued functions of one real variable.

1 The set of real numbers

1.1 Subsets of $\mathbb{R}$

Example 1. What is a number ?

Definition 1.
We distinguish several subsets of real numbers.
- The set of natural numbers or positive integers, denoted by $\mathbb{N}$.
- The set of integers, denoted by $\mathbb{Z}$.
- The set of decimal numbers (its decimal expansion terminates after a finite number of digits),denoted by $\mathbb{D}$.
- The set of rational numbers (Quotient of two integers), denoted by $\mathbb{Q}$. The decimal expansion of a rational number either terminates after a finite number of digits or begins to repeat the same sequence of digits over and over.
- The set of irrational numbers (it means real numbers which are not rational).

1.2 The absolute value

Definition 2.
Let x be a real number, its absolute value, denoted by $|x|$, is defined as follows :
$|x| = x$ if x is positive, $|x| = -x$ if not.

Example 2.
Express without the absolute value $\left|\dfrac{\sqrt{2}-1}{\sqrt{2}-3}\right|$.

Property 1.

- $|x| \geqslant 0$

- $|-x| = |x|$
- Triangular Inequality $||a| - |b|| \leqslant |a+b| \leqslant |a| + |b|$
- $\sqrt{x^2} = |x|$

Example 3.
Find examples where the above inequality is strict, and where the above inequality is an equality.

There is a link between absolute values and real intervals.

Property 2.
Let x, a and ε be real numbers.
- $|x-a| = \text{distance}(a;x)$
- $|x-a| \leqslant \varepsilon \iff x \in [a-\varepsilon; a+\varepsilon]$

Example 4.
Draw on the real line the previous properties.

2 Domains of definition.

2.1 In mathematics.

In mathematics, a function is an operator between two sets. However, in practice, we often give a mathematical expression to define a function, without specifying its starting set. The **domain of definition** of a function defined by its expression is the set of real numbers for which the expression has a mathematical meaning (i.e. that all the operations that one needs to do to compute the expression of the function are allowed).

Example 5.
Give the domain of definition of the functions f and g defined by

$$f(x) = \frac{x^2+2x-3}{x^2-1}, \quad g(x) = \frac{\ln(x)}{x-2}.$$

2.2 A physical example : the force of gravitational interaction.

In Newton's theory, the force of gravitational interaction between two masses m_A and m_B is proportional to the product of the two masses and inversely proportional to the square of the distance between them d :

$$F_{AB} = G\frac{m_A m_B}{d^2}$$

and G is the gravitational constant.

2.2.1 Variables and functions

In our previous examples, there exists several letters. Those can be interpreted differently.
We may use this formula to compute the force knowing others values.

Example 6.
Compute the force of gravitational interaction between two protons knowing that $m_A = m_B = 1,67\ 10^{-27}$kg, $G = 6,67\ 10^{-11}$ N.m^2.kg^{-2} et $d = 2,32\ 10^{-15}$ m.

Indeed, F depends on three inputs. In mathematics, we say that F is a **function** of three **variables** : m_A, m_B et d.

We may also fix two of those three inputs, for instance m_A et m_B. F will be a function of one variable d. m_A and m_B are called **constants**.

Example 7.
In mathematics, how do we denote the image of d under F ?

2.2.2 Domain of definition

We want to study F depending on d, thus we have to know what the set of real numbers such that $F(d)$ is defined. This set is called **domain of definition** of the function F. In our example, d can take any strictly positive value, thus the domain of definition of F is $]0; +\infty[$.

Remark 1.
Just looking at the definition $F(x) = G\dfrac{m_A m_B}{x^2}$, the domain of definition of F would be $\mathbb{R}^*$. The domain of definition of a function is different wheter we consider the physical or the mathematical context of the function.

2.2.3 Limits of F when the variable approaches a real number

We can't compute $F(d)$ if $d = 0$, but we can compute $F(d)$ for d very close to 0. In mathematics, d can be as close as we want to 0.

Example 8.
What can you say about $F(d)$ when d approaches, as close as we want, 0 ? What does it mean in physics ?

2.2.4 Limit of F when the variable approaches infinity $+\infty$

We can compute $F(d)$ when d takes values as big as we want.

Example 9.
What can you say about $F(d)$ when d takes values as big as we want ? What does it mean in physics ?

3 Operations on limits.

The precise definition of a limit will be given formally in a later section. Let's first see what are the usual operations that we are allowed to do in limit computations. In the following, a is either a real number or infinite.

3.1 Limit of a sum

$\lim\limits_{x \to a} (f(x) + g(x)) =$

$\lim_{x\to a} g(x) =$ \ $\lim_{x\to a} f(x) =$	$l \in \mathbb{R}$	$+\infty$	$-\infty$
$m \in \mathbb{R}$			
$+\infty$			
$-\infty$			

3.2 Limit of a product

$\lim_{x\to a} (f(x).g(x)) =$

$\lim_{x\to a} g(x) =$ \ $\lim_{x\to a} f(x) =$	$l > 0$	$l < 0$	0	$+\infty$	$-\infty$
$m > 0$					
$m < 0$					
0					
$+\infty$					
$-\infty$					

3.3 Limit of a quotient

$\lim_{x\to a} \frac{f(x)}{g(x)} =$

$\lim_{x\to a} g(x) =$ \ $\lim_{x\to a} f(x) =$	$l > 0$	$l < 0$	0	$+\infty$	$-\infty$
$m > 0$					
$m < 0$					
0^+					
0^-					
$\pm\infty$					

Example 10.
Give an example of an indeterminate form in each previous case.

Example 11. For each of the following examples, say if we can conclude directly or not on the existence of a limit.

1. $\sqrt{x+2} - \sin(x)$ when $x \to 0$?
2. $\frac{\sqrt{x^2+2}}{2x+7}$ when $x \to +\infty$?
3. $\frac{e^{-x}}{x^2}$ when $x \to 0$? When $x \to +\infty$?

0. IF= Indeterminate Form, all cases are possible, we may not predict what happens

4. $x - \sqrt{x^2+7}$ when $x \to +\infty$? When $x \to -\infty$?

3.4 Limits of composition of functions

Definition 3.
Let's assume that f is a unfction defined on a set I and g a function defined on $f(I)$. Then $g \circ f$ is the function defined on I by $g \circ f(x) = g(f(x))$ pour tout $x \in I$.

Example 12.

1. Give the expression of $f \circ g$ and $g \circ f$ with $f(x) = x + 3$ et $g(x) = x^2 - 1$.
2. Find two functions f and g such that $h = f \circ g$ with $h(x) = \sqrt{x^4 + 2x^2 + 1}$.

Theorem 1.
Let's consider $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ such that $f(I) \subset J$ thus the composition $g \circ f$ exists.

1. If f has a limit (finite or not) b at the point a then b is an element or a endpoint of J
2. Moreover if g has a limit (finite or not) l at the point b, then $g \circ f$ has the limit l at the point a :

$$\begin{cases} f(x) \underset{x \to a}{\to} b \\ g(x) \underset{x \to b}{\to} l \end{cases} \Rightarrow g\left(f(x)\right) \underset{x \to a}{\to} l$$

Example 13.
Compute $\lim_{x \to +\infty} \ln\left(1 + \frac{1}{x}\right)$

3.5 Limits at infinity for polynomial and rational functions

Theorem 2.
The limit at infinity $\pm\infty$ of a polynomial function P, which associates to each x a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, is equal to the limit at $\pm\infty$ of the term of higher degree of P : $a_n x^n$.
Thus we have :

$$\lim_{x \to \pm\infty} a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \lim_{x \to \pm\infty} a_n x^n$$

Example 14. Give the limit of $f(x) = 2x^2 + 3x - 1$ at $-\infty$.

Theorem 3.

There is a basic rule for evaluating limits at infinity for a rational function Q defined as the quotient of two polynomials : $Q(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$.
The limit at infinity $\pm\infty$ of Q is equal to the quotient of terms of higher degrees at the numerator over the terms of higher degrees at the denominator.
Thus we have :

$$\lim_{x \to \pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$$

Example 15.
Find the limits of $f(x) = \dfrac{2x^2+3x-1}{5x^2-x+3}$ at $+\infty$ and $g(x) = \dfrac{2x^2+3x-1}{7x^3+x^2+3}$ at $-\infty$.

3.6 Limits and square root functions

There exists two main examples :

- Indeterminate form $+\infty$ of type $\dfrac{\sqrt{x+3}}{\sqrt{x-4}}$: we factorize by $\sqrt{x}$ at the numerator and at the denominator.
- Indeterminate form of type $\sqrt{x+3}-\sqrt{x+6}$: we multiply and divide by the "conjugate" $\sqrt{x+3}+\sqrt{x+6}$.

Example 16.
Give the limits at $+\infty$ for the two previous examples.

4 Limits and Inequalities

4.1 Theorem

a is either a real number or infinite.

Theorem 4. *Limits and inequalities theorem*
Let f, g be two real valued functions defined on I. We assume that both f and g have finite limits l and m at some point a and that $f \leqslant g$ at the neighborhood of a then : $l \leqslant m$

Example 17.
If f, g are two real-valued functions defined I such that f and g have both finite limits l and m at the point a and such that $f < g$ at the neighborhood of a do we have : $l < m$?

Theorem 5. *The sandwich theorem (or squeeze theorem or pitching theorem*

Let f, g, h be three real valued functions defined on I. Let's assume that : $\begin{cases} f(x) \underset{x\to a}{\to} l \\ h(x) \underset{x\to a}{\to} l \end{cases}$ and $f \leqslant g \leqslant h$ at the neighborhood of a, then it follows $g(x) \underset{x\to a}{\to} l$

Example 18. Give the limit at $+\infty$ of $\dfrac{\cos x}{x}$

Remark 2. (Case $l = 0$) In particular, the previous theorem implies that $f(x) \underset{x\to a}{\to} 0$ if, and only if, $|f(x)| \underset{x\to a}{\to} 0$.

Property 3.

1. Let f, g be two real-valued functions defined on I. Let's assume that $f(x) \underset{x\to a}{\to} +\infty$ and $f \leqslant g$ at the neighborhood of a, then it follows $g(x) \underset{x\to a}{\to} +\infty$

2. Let f, g be two-real valued functions defined on I. Let's assume that $g(x) \underset{x\to a}{\longrightarrow} -\infty$ and $f \leqslant g$ at the neighborhood of a, then $f(x) \underset{x\to a}{\longrightarrow} -\infty$

Example 19.
Let's define $f : \mathbb{R} \to \mathbb{R}, x \mapsto x + \cos x$. What are : $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$?

5 Limit of a monotonic function

Theorem 6.
Let f be a function defined on an open interval $I =]a; b[$, with a and b in $\mathbb{R} \cup \{-\infty; +\infty\}$. Let's assume that f is a monotonic function on I, then f has a finite or infinite limit at the point a and at the point b.

Example 20.

1. What can you say about the limit of a function f in each case :
(a) f is an increasing function ?
(b) f is also bounded above ?
2. Give necessary condition(s) on f so that f admits a finite limit ? Are those conditions sufficient ?

6 Logic Quantifiers

6.1 Universal Quantifier $\forall$

"$\forall x \in E$" means :" **for all** element x in E".

Example 21.
Write using quantifiers : " the square of a real number is always positive".

6.2 Existential Quantifier $\exists$ and $\exists!$

- "$\exists x \in E$" means : " **it exists** at least an element x in E".
- " / " or " , " or " | " or " ;" means : " such that".
- "$\exists!$", means there exists one and only one element ...(uniqueness quantifier), a unique element...

Example 22.
Write using mathematical quantifiers :
" The equation $2x^2 - x = 0$ has an integer solution."
" 1 has a unique inverse image under the map ln."

6.3 Negating quantifiers

Theorem 7. The rules for negating quantifiers are

1. not $(\forall x \in E \quad \mathrm{P}(x)) \quad \Leftrightarrow \quad (\exists x \in E/ \text{ not } \mathrm{P}(x))$
2. not $(\exists x \in E/\mathrm{P}(x)) \quad \Leftrightarrow \quad (\forall x \in E \quad \text{not } \mathrm{P}(x))$

Example 23.
Express the negation of the following sentences :
P : $\exists x \in \mathbb{R} : f(x) = 3$.
Q : $\forall y \in \mathbb{R}$ $f(y)$ is an integer
R : $\forall y \in F \quad \exists x \in E \; / \; f(x) = y$

7 Definition of a limit

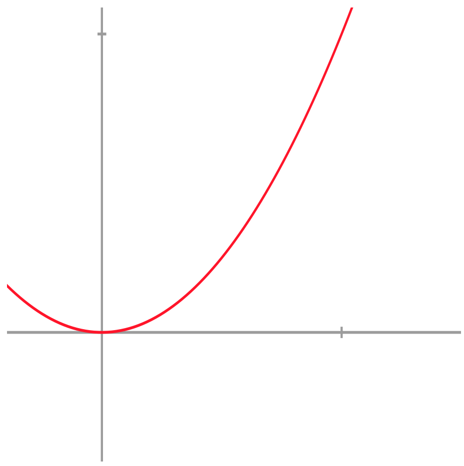
In this section, f is a real-valued function of one variable, defined on an interval I.
$a \in \mathbb{R} \cup \{-\infty, +\infty\}$

7.1 Finite limit at the point a

Definition 4.
Let $l \in \mathbb{R}$.
Notation : $\lim\limits_{x \to a} f(x) = l$ or $f(x) \underset{x \to a}{\longrightarrow} l$, is read : "$f$ has the limit l at the point a".

1. First Case $a \in \mathbb{R}$: the limit of f as x approaches a is l if and only if :

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in I \backslash \{a\}, |x - a| \leqslant \alpha \Rightarrow |f(x) - l| \leqslant \varepsilon$$



Example 24. Write, using quantifiers, that $\lim\limits_{x \to a} f(x) \neq l$.

2. Second Case $a = +\infty$: the limit of f as x approaches $+\infty$ is l if and only if :

$$\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in I, x \geqslant A \Rightarrow |f(x) - l| \leqslant \varepsilon$$

3. Third Case $a = -\infty$: the limit of f as x approaches $-\infty$ is l if and only if :

$$\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in I, x \leqslant A \Rightarrow |f(x) - l| \leqslant \varepsilon$$

Intuitively, $\lim_{x \to a} f(x) = l$ means : whatever is $\varepsilon > 0$, to get $f(x)$ close enough to l with the term of error equal to $\leqslant \varepsilon$, it is sufficient to take x sufficiently close to a, in a neighborhood of a. Be careful as the neighborhood of a depends on ε.

Example 25. Show that if $f(x) \underset{x \to +\infty}{\to} 1$, then for all x large enough, $f(x) \geqslant 0$.

Theorem 8. *Uniqueness of the limit*
Let's consider $f : I \to \mathbb{R}$ a function and l, l' two real numbers. Let's assume that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = l'$. Then $l = l'$.

Remark 3. The limit at the point a is unique but is not always equal to $f(a)$.

7.2 Infinite limit at the point a

In this section, a is a endpoint (finite or not) of the interval I.

Definition 5.
Let $f : I \to \mathbb{R}$ be a function.

1. First Case $a \in \mathbb{R}$: the limit of f as x approaches the point a is infinity $+\infty$ (respectively $-\infty$) if and only if :

$$\forall B \in \mathbb{R}, \exists \alpha > 0, \forall x \in I \backslash \{a\}, |x-a| \leqslant \alpha \Rightarrow f(x) \geqslant B \text{ (respectivement, f(x)} \leqslant \text{B)}$$

2. Second Case $a = +\infty$: the limit of f as x approaches $+\infty$ is is infinity $+\infty$ (respectively $-\infty$) if and only if :

$$\forall B \in \mathbb{R}, \exists A \in \mathbb{R}, \forall x \in I, x \geqslant A \Rightarrow f(x) \geqslant B \text{ (respectively, f(x)} \leqslant \text{B)}$$

3. Third Case $a = -\infty$: the limit of f as x approaches $-\infty$ is $+\infty$ (respectively $-\infty$) if and only if :

$$\forall B \in \mathbb{R}, \exists A \in \mathbb{R}, \forall x \in I, x \leqslant A \Rightarrow f(x) \geqslant B \text{ (respectively, f(x)} \leqslant \text{B)}$$

We denote it : $\lim\limits_{x \to a} f(x) = +\infty$ respectively $\lim\limits_{x \to a} f(x) = -\infty$

Example 26.
Let f be a function defined on $\mathbb{R}$ and a be a real number. Draw a function checking this property :

$$\exists \alpha > 0 : \forall \varepsilon > 0, |x-a| < \alpha \Longrightarrow |f(x) - l| \leqslant \varepsilon$$

7.2.1 Left and right hand limits

Definition 6. *Left hand limit*
Let's assume that x_0 is not the left endpoint of I. Studying the left hand limit of f at the point x_0 means that we study the values of f only for $x < x_0$ and $x \in I$. We denote it : $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x)$ or $\lim\limits_{x \to x_0^-} f(x)$

Example 27. Give $\lim\limits_{x \to 0^-} \dfrac{1}{x}$

Definition 7. *Right hand limit*
Let's assume that x_0 is not the right endpoint of I.Studying the right hand limit of f at the point x_0 means that we study the values of f only for $x > x_0$ and $x \in I$. We denote it : $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x)$ or $\lim\limits_{x \to x_0^+} f(x)$.

Example 28. Give $\lim\limits_{x \to 0^+} \dfrac{1}{x}$

Property 4. We assume that x_0 is an element of I but is not one of its endpoints. f has a limit at the point x_0 if and only if the right hand limit is equal to the left hand limit. The limit of f at the point x_0 is the common value.

Remark 4.
If x_0 is one of the endpoints of I then :

- Either $I =]x_0; ...$ or $I = [x_0; ..$ then $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0^+} f(x)$.
- Either $I = ...; x_0[$ or $I = ...; x_0]$ then $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0^-} f(x)$.

Example 29. Let's define : $\delta : \begin{array}{c} \mathbb{R} \to \mathbb{R} \\ x \mapsto \begin{cases} 1 \text{ si } x = 0 \\ \dfrac{|x|}{x} \text{ si } x \neq 0 \end{cases} \end{array}$

Does δ have a limit at 0 ?

Example 30. Does the following function have a limit at 0 ? $f : \begin{array}{c} \mathbb{R} \to \mathbb{R} \\ x \mapsto \begin{cases} \sqrt{x}, \text{ si } x \geqslant 0 \\ -x \text{ si } x < 0 \end{cases} \end{array}$

8 Continuity

8.1 Continuous Function

We study the limit of a function at some point of its domain of definition

Definition 8.
Let $f : I \to \mathbb{R}$ be a function.

1. Let $x_0 \in I$. f is continuous at x_0 means that :
   - f has a finite limit at x_0
   - $\lim\limits_{x \to x_0} f(x) = f(x_0)$

   If one of those condition is not checked, f is said discontinuous at x_0.
2. f is continuous on I if and only if this function is continuous at all points x_0 of I.

Example 31.
Is the function δ previously defined 29 continuous at 0 ?

Property 5.
If f is continuous at a and if g is continuous at $f(a)$ then $g \circ f$ is continuous at a.

Example 32.
Let's define f by $f(x) = \ln(\sqrt{x} + 1)$. Is f continuous at 0 ?

Property 6.
Let's assume that f is a function defined as operations (sum, product...) of usual functions (sin called the sine, cos called the cosine, rational functions...). then f is continuous on its domain of definition.

Remark 5.
To prove that a function is continuous at some points we either use the previous proposition (useful for the points where f coincides with usual functions), or we use the 8.

Example 33.
Let f be the fonction defined on $\mathbb{R}$ by :

$$\begin{cases} f(0) = 2 \\ f(x) = \dfrac{x^2 + x}{x} \end{cases}$$

Is this function$f$ continuous on $\mathbb{R}$?

8.2 Continuous Extension

Definition 9.
Let's consider f a function defined on an open interval I, and let a be one of its endpoints. If f has a limit l at a, we **extend f by continuity at** a setting $f(a) = l$.

Example 34.
Let's define f on $]0; +\infty[$ by :

$$f(x) = \frac{x}{\sqrt{x}}$$

Is it possible to extend f at 0 ?

8.3 Image of an interval through a continuous function

Definition 10.
Let I be a subset of $\mathbb{R}$. The image of I under f is the set of real numbers $f(x)$ where $x \in I$. We denote it $f(I)$ so $f(I) = \{f(x)/x \in I\}$

Example 35.
Give $f(I)$ if $I = [-\sqrt{5}; 2, 5]$ and f is the floor function.

Example 36.
Give $f(I)$ if $I = [-1; 3]$ and $f : x \mapsto x^2$.

Property 7.
The image of an interval under a continuous function is an interval.

Example 37.
Give the image of $\left[\frac{\pi}{4}; \frac{7\pi}{4}\right]$ under the sine.

Property 8.
The image of $[a, b]$ under a continuous and increasing function f is $[f(a); f(b)]$.
The image of $]a, b]$ under a continuous and decreasing function f is $[f(b); f(a)[$.

Example 38.
Show that if f is continuous but not monotonous, the previous result doesn't work anymore.

Remark 6.
The previous property can be applied for open, closed, left-opened, right-opened intervals.

Remark 7. The previous property allows us to determine the image of I by f by studying the variations of the function.

Example 39. Compute $f(I)$ using the variation table of the function for the two previous examples.

Exercise 1.
Develop the following expressions :

1. $(x-1)(x+2)-x(x-1)$
2. $(a-b)(a+b)$
3. $(-3x-1)^2$

Exercise 2.
Factorise the following expressions :

1. $2x^3-5x^2$
2. $x^2-1+(3x-2)(x+1)$
3. $(x-1)(3x+2)+(5x-3)(-x+1)$
4. $2np-px+p$
5. $3a^2b+ab-4ab^2$
6. x^n+nx^{2n}
7. 2^n+8^n

Exercise 3.
Reduce the following fractions and find their domain of definition :

1. $\dfrac{3x+2}{6x+4}$
2. $\dfrac{x^3+2x+x}{x}$
3. $\dfrac{2+3b}{2}$
4. $\dfrac{2ab+b}{3b-4ab}$
5. $\dfrac{\frac{x+2}{x+3}}{\frac{x-1}{5x+2}}$
6. $\dfrac{\frac{3a-1}{2a+1}+2}{\frac{2+a}{3-a}+3}$

Exercise 4.
Simplify the following expressions :

1. $\sqrt{9x+9}$
2. $\sqrt{2^5}$
3. $\dfrac{1}{\sqrt{\frac{x+2}{x-1}}}$

Exercise 5.
Find the real numbers x such that :

1. $|x-2|<3$
2. $|x+4|<-1$
3. $|x+5|\leqslant 3$
4. $|x+2|\geqslant 5$
5. $|x+5|\geqslant -1$
6. $|x-2|<5$ and $|x-3|\leqslant 4$
7. $|x+1|<1$ or $|x+2|\leqslant 4$
8. $|x-5|<2$ and $|x+1|\leqslant 1$
9. $|x-2|<4$ or $|x+2|\leqslant 2$

Exercise 6.
Let a and b two real numbers, find the real numbers α and β (depending on a and b) such that $x\in[a;b] \iff |x-\alpha|\leqslant\beta$.

Exercise 7.
Find the points $M(x;y)$ such that :

1. $|x+y| = 2$
2. $|x+y| \leqslant 2$
3. $|x| + |y| \leqslant 3$

Exercise 8.
Give the domain of definition of the following functions :

1. $f(x) = \dfrac{1}{x+2}$;
2. $f(x) = \dfrac{\sin x}{x}$
3. $f(x) = \sqrt{-2x+3}$

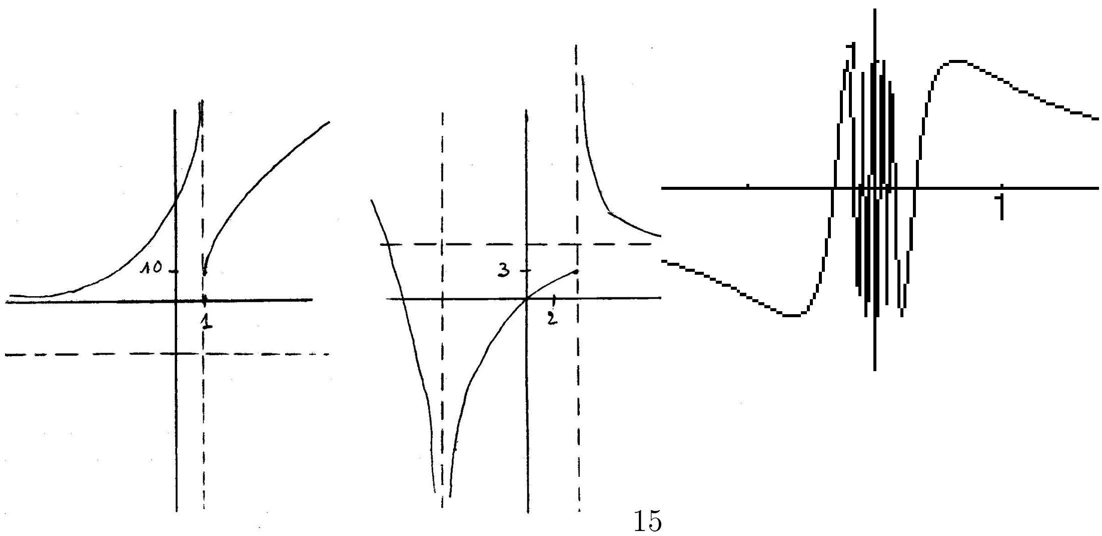
Exercise 9.
Give the mathematical notation for the functions below (f is the function and x the variable), then give the domain of definition.

1. $H = \dfrac{r+\omega}{\lambda\omega(1-\omega^2)}$ (H is called transfer function).
2. $z = \dfrac{g}{2}t^2$ (z is the distance travelled by a ball in a freefall).
3. $P = \dfrac{nRT}{V}$ (P is the thermodynamic pressure).
4. $F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{r^2}$ (F is the electrostatic field between charges).

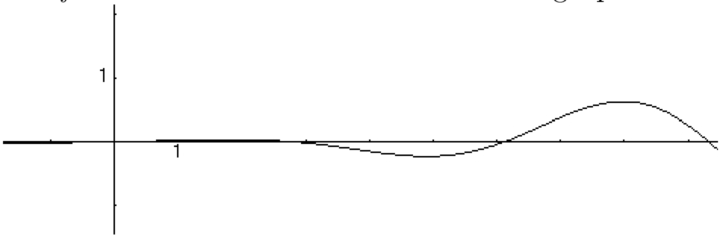
Exercise 10.
In each case, draw the shape of the curve.

1. $\lim\limits_{2^+} f = 3$, $\lim\limits_{2^-} f = -\infty$, $\lim\limits_{+\infty} f = 5$ et $\lim\limits_{-\infty} f = +\infty$.
2. $\lim\limits_{-1^+} f = -\infty$, $\lim\limits_{-1^-} f = 0$, $\lim\limits_{-\infty} f = 0$ et $\lim\limits_{+\infty} f = -\infty$.

Exercise 11.
In the following examples, conjecture the limits of f ? Explain why this is just a conjecture.



Exercise 12.
Let f be the fonction associated to this graph :



1. Let's predict the limit of f in 0.
2. We have $f : f(x) = 10^{-2}(10 + x^2)\sin\dfrac{1 + x^2}{x}$.
   Draw this function with you calculator. Do you get the same curve ?
3. Using the expression of f, draw the representation of f for $x \in [-1; 1]$.
4. Draw the previous curve on your calculator.
5. Is a graph reliable to find limits ?

Exercise 13.

1. Study the limit at $+\infty$ of the two following quantities :

$$a)\ \frac{3x+7}{6x-13} + \frac{14x^2+2}{7x^2+31x-4} - 5 + \frac{1}{3x+4}, \qquad b)\ x + 2 + \left(\frac{\sin(x) - 3 + \ln(x)}{4 + \sqrt{x}}\right)^2.$$

2. Study the limits at $+\infty$ and at $-\infty$ of the expression

$$\frac{e^x + 1 + \frac{1}{x}}{e^x + 4}.$$

3. For each case, compute the following limits if they exist :

$$a)\ \lim_{x\to 0} \frac{x^2 + 2\,|x|}{x} \qquad b)\ \lim_{x\to -\infty} \frac{x^2 + 2\,|x|}{x} \qquad c)\ \lim_{x\to 2} \frac{x^2 - 4}{x^2 - 3\,x + 2}$$

$$d)\ \lim_{x\to \pi} \frac{1 - \cos^2 x}{1 + \cos x} \qquad e)\ \lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1+x^2}}{x} \qquad f)\ \lim_{x\to +\infty} \sqrt{x+5} - \sqrt{x-3}$$

Exercise 14.
Find the limits of the following functions.

1. $\lim\limits_{t\to+\infty} \dfrac{Rt^2 + C}{R^2\omega t^2 + C}$
2. $\lim\limits_{t\to 0} \dfrac{Rt^2 + C}{R^2\omega t^2 + C}$
3. $\lim\limits_{t\to+\infty} N_0 e^{-kt}$.

Exercise 15.
Find the limits of the following functions.

1. $\lim_{x \to +\infty} -2x - 3\cos x$
2. $\lim_{x \to -\infty} x + \sin x$
3. $\lim_{x \to +\infty} \frac{x + \cos x}{x - \sin x}$

Exercise 16.
In each case, draw a function checking the given property :

1. $\exists l \in \mathbb{R} : \forall (a, b) \in \mathbb{R}^2 \exists x \in [a, b] : f(x) = l$
2. $\forall (a, b) \in \mathbb{R}^2 \exists l \in \mathbb{R}, \exists x \in [a, b] : f(x) = l$

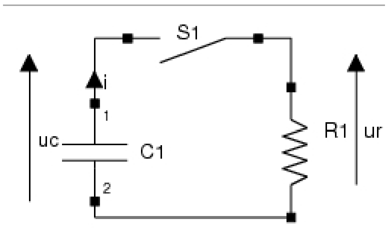
Exercise 17.
Write, using mathematical symbols, that a function has no finit limit at infinity.

Exercise 18.
Are the following functions continuous at a ?

1. $f(x) = \frac{x}{\sqrt{|x|}}$ for $x \neq 0$ and $f(0) = 1$, $a = 0$.
2. $f(x) = \frac{x^2 + x - 2}{x^2 + 3x - 4}$ for $x \neq 1$ and $x \neq -4$ and $f(1) = 10$, $a = 1$.

Exercise 19.
Find an example of a function f, such that f has a limit in a, but is not continuous at a.

Exercise 20.
We consider the electrical circuit below.



When this circuit is closed, we have $U_c = Ee^{-\frac{t}{\tau}}$ and $U_r = U_c$ with τ a positive constant.

1. Give the domain of definition of U_c and of U_r.
2. Study the continuity of those functions on the previous sets.
3. Find the limit of U_C when t tends to $+\infty$ and give a physical interpretation.

Exercise 21.
Is it possible to extend by continuity the following functions to the specified points ?

1. $f(x) = \frac{x^2 - 4}{x - 2}$, $x = 2$
2. $f(x) = \frac{x^2 - 9}{\sqrt{x + 3}}$, $x = -3$

3. $f(x) = \dfrac{\sqrt{3x^2+1}-2}{x-1}$, $x = 1$

Exercise 22.

1. Give the image of the set]-5 ;0] through the function f defined by $f(x) = \dfrac{1}{x-1}$.
2. Solve in $\mathbb{R}$ $-3 \leqslant x^2 < 7$.
3. Give the set of real numbers I such that $f(I) = J$, with the function f $f(x) = e^{-x^2}$ and $J = [-2; 5]$.

Exercise 23.
Let f be a fonction defined on $\mathbb{R}$.

1. Can we have both f continuous, strictly increasing f on the set $[a; b]$ and $f([a; b]) = [f(a); f(b)]$?
2. Is is possible to have a function f not continuous in $c \in]a; b[$, strictly increasing on the set $[a; b]$ and such that
$f([a; b]) = [f(a); f(b)]$?
3. Give the reciprocal of the line 1 of property 8.
4. Is the reciprocal true ? ?

Exercise 24.
Let f be defined by $f(x) = (-1)^{E(x)}(x - E(x))$ where $E(x)$ denotes the floor function which associates to x the largest integer less than or equal to x. Study the continuity of f.