Calculus and Logic

- Understand sets of numbers
- Master quantifiers and types of reasoning
- Know how to solve equations and inequalities
- Know how to solve a linear system of two equations with two unknowns

1 Sets of numbers

1.1 Subsets of $\mathbb{R}$

Example 1. What is a number ?

Definition 1.
Several subsets of the set of real numbers are distinguished :
- The set of natural numbers, denoted by $\mathbb{N}$.
- The set of integers, denoted by $\mathbb{Z}$.
- The set of decimal numbers (numbers with a finite number of digits after the decimal point), denoted by $\mathbb{D}$.
- The set of rational numbers (quotients of two integers), denoted by $\mathbb{Q}$. The decimal part of a rational number is periodic from a certain point onwards.
- The set of irrational numbers (real numbers that are not rational).

Regarding notation, it is important to distinguish between the symbol $\in$, which relates an element to a set (e.g., $1 \in \mathbb{N}$), and the inclusion symbol $\subset$, which relates sets to one another (e.g., $\mathbb{N} \subset \mathbb{Z}$).

1.2 Complex numbers

Definition 2. A complex number is of the form $a+ib$, where a and b are real numbers and i is such that $i^2 = -1$. This representation is called the algebraic form of the complex number ; a is called the real part and b the imaginary part. The set of complex numbers is denoted by $\mathbb{C}$.

Example 2. State the real and imaginary parts of the complex numbers $z_1 = 3 + 2i$ and $z_2 = 1 - 2i$.

Definition 3. The complex number $a - ib$ is called the conjugate of the complex number z and is denoted by $\bar{z}$.

Example 3. Give the conjugate of $z = 1 - i$.

Proposition 1. Let $z_1 = a+ib$ and $z_2 = c+id$; then addition, multiplication, and the quotient are defined as follows :

$$z_1 + z_2 = (a + c) + i(b + d)$$

$$z_1 * z_2 = (ac - bd) + i(ad + bc)$$

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(a+ib)(c-id)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}.$$

Example 4. Calculate z_1+z_2, z_1z_2, and $\dfrac{z_1}{z_2}$ for the complex numbers from the previous example.

Example 5. Let $z = -1+i$; calculate z^2 and show that $z^2+2z+2=0$.

2 Mathematical notation : implication, equivalence, quantifiers

2.1 Proposition, AND, OR

Definition 4.

- A proposition (or assertion) is a mathematical statement that has one and only one truth value : true or false.
- The negation of the proposition P is the proposition that is true if and only if P is false. It is denoted by $\neg P$ (or not(P)).
- If P and Q are two propositions, $P \wedge Q$ (or P and Q) is the proposition that is true if and only if both P and Q are true.
- If P and Q are two propositions, $P \vee Q$ (or P or Q) is the proposition that is true if and only if at least one of the two propositions, P or Q, is true.
- The operators $\neg$ (not), $\wedge$ (and), and $\vee$ (or) are related by the following formulas :

$$\neg(P \wedge Q) = (\neg P) \vee (\neg Q).$$

$$\neg(P \vee Q) = (\neg P) \wedge (\neg Q).$$

Example 6. State whether the following propositions are true or false :

1. 136 is a multiple of 17 and 2 divides 167.
2. 136 is a multiple of 17 or 2 divides 167.

2.2 Implication and equivalence.

If P and Q are two propositions, the implication $P \Longrightarrow Q$ is the proposition not$(P) \vee Q$. To prove $P \Longrightarrow Q$, one assumes that P is true and proves that Q is true.

Remarque 1. In mathematics, if P is false, then $P \Longrightarrow Q$ is true.

When $P \Longrightarrow Q$, we say that P is a **sufficient condition** for Q and that Q is a **necessary condition** for P.

Example 7.

1. Find a condition that is sufficient but not necessary for the proposition A : "$x^2 = 9$", and for B : "$x > 1$".
2. Find a condition that is necessary but not sufficient for the proposition A : "$x^2 = 9$", and for B : "$x > 1$".

Propositions P and Q are said to be **equivalent** if $P \Longrightarrow Q$ and $Q \Longrightarrow P$; in this case, we write $P \iff Q$.

Example 8. Solve the equation (E) : $\quad x = \sqrt{2-x}$.

2.3 Quantifiers.

2.3.1 Universal quantifier "$\forall$".

"$\forall x \in E$" means : "**for all** elements x of E".

Example 9.
Write using mathematical symbols : "the square of a real number is always positive".

2.3.2 Existential quantifier : "$\exists$" and "$\exists !$".

- "$\exists x \in E$" means : "**there exists** at least one element x of E" satisfying the remainder of the proposition.
- " / ", " : ", ",", "|" or a space mean : "such that".
- "$\exists !$" means there exists a **unique**....

Example 10.
Write using only mathematical symbols :
"The equation $2x^2 - x = 0$ has an integer solution."
"1 has a unique preimage under the function ln."

2.3.3 Négation des quantificateurs

Theorem 1.

1. not $(\forall x \in E \quad \mathrm{P}(x)) \quad \Leftrightarrow \quad (\exists x \in E / \text{ not } \mathrm{P}(x))$
2. not $(\exists x \in E / \mathrm{P}(x)) \quad \Leftrightarrow \quad (\forall x \in E \quad \text{not } \mathrm{P}(x))$

Example 11.
Write the negation of the following propositions :
P : $\exists x \in \mathbb{R} : f(x) = 3$.
Q : $\forall y \in \mathbb{R}\ f(y)$ est un entier
R : $\forall y \in F \quad \exists x \in E \ / \ f(x) = y$
S : $\exists \alpha > 0 \, ; \, x \in]-\alpha, \alpha[\implies f(x) \in \left]-\frac{1}{10}, \frac{1}{10}\right[$.

3 Principles of mathematical reasoning

3.1 Proof by induction

- Verify that the property holds for $n = n_0$: this is the base case.
- Assume that the property holds for a given integer $n \geqslant n_0$ (this is the inductive hypothesis) and prove, using this hypothesis, that the property holds for $n+1$: this is the inductive step.
- Conclude that the property holds for all integers $n \geqslant n_0$.

Example 12. Prove that for all $n \geqslant 1$,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

3.2 Proof by contradiction

Suppose that in a certain context we wish to prove a proposition P. Proof by contradiction consists of assuming that P is false and then constructing a proof that yields a contradiction—that is, a proposition Q such that both Q and "not Q" are true (often Q is P, though not always). The law of excluded middle then leads us to reject the assumption that "P is false," thereby establishing that P is true.

Example 13. Prove that 0 has no inverse.

3.3 Proof by contrapositive

Given two propositions P and Q, proving $P \implies Q$ is equivalent to proving the implication known as the contrapositive : $\text{not}(Q) \implies \text{not}(P)$.

Example 14. Let m be a natural number. Prove that if m^2 is even, then m is even.

4 Simplifications

4.1 Notable identities

Definition 5. Factorials
The factorial of a natural number n, denoted by $n!$, is defined as follows :

$$0! = 1 \quad \text{by convention,}$$

$$n! = 1 \times 2 \times 3 \times \cdots \times n = \prod_{k=1}^{n} k \quad \text{for all } n \geqslant 1.$$

If $n \in \mathbb{Z}$ and $n < 0$, then $n! = 0$.

Example 15. Calculate 2!, 3!, 4!.

Proposition 2. For $n \in \mathbb{N}$,

$$n! = (n-1)! \times n$$

.

Example 16. Simplify the following expressions : $\dfrac{n!}{(n-1)!}$; $\dfrac{(n-1)!}{(n+1)!}$

Definition 6. Binomial coefficients
For integers k and n such that $0 \leqslant k \leqslant n$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k < 0$ or $k > n$, $\binom{n}{k} = 0$.

Example 17. Calculate $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}$

Proposition 3. For any pair (k, n), the following relation holds :

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This allows for the construction of **Pascal's Triangle**.

Example 18. Find $\binom{5}{k}$ for $0 \leqslant k \leqslant 5$.

Proposition 4. Let a and b be two real numbers.

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$a^2 - b^2 = (a-b)(a+b)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Example 19.

Develope $(2x-1)^3$.
Factorise the expression : $x^2 - 9 + (2x-3)(x-3)$.

4.2 Fractions

Proposition 5.

- addition :$\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$
- multiplication : $\dfrac{a}{b} * \dfrac{c}{d} = \dfrac{ac}{bd}$
- division : $\dfrac{a}{b} / \dfrac{c}{d} = \dfrac{ad}{bc}$

Example 20. Simplify $\dfrac{\frac{11}{4} - \frac{1}{2}}{\frac{7}{3} + \frac{5}{6}}$

4.3 Powers

Proposition 6. $n \in \mathbb{N} \quad p \in \mathbb{N}$

$$a^n * a^p = a^{n+p}$$

$$(a^n)^p = a^{np}$$

$$\frac{1}{a^n} = a^{-n}$$

$$a^0 = 1 \quad a^1 = a$$

$$a^n * b^n = (ab)^n$$

$$\frac{a^n}{b^n} = (\frac{a}{b})^n$$

$$\forall a \geqslant 0 \quad \sqrt{a} = a^{1/2}$$

Example 21. After stating their domains of definition, simplify the following expressions :

$$\frac{\sqrt{x}}{x}; \quad \frac{x}{\sqrt{-x}}; \quad (x+1).\left(\frac{1}{x+1}\right)^{-3}.$$

5 Solving equations and inequalities.

5.1 Polynomial equations.

General concepts.

Definition 7. A **polynomial function** (or, loosely speaking, a **polynomial**) is an expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, where n is an integer and the a_k are real coefficients. The **degree** of a polynomial is the index of the non-zero coefficient associated with the highest power of x (i.e., x^k).
A polynomial equation is an equation of the form $P(x) = c$, where P is a polynomial function and c is a real number.

Example 22. $P(x) = -x^4 + 3x - 2$ is a polynomial of degree 4 ; $Q(x) = 2x + 3x^2$ is a polynomial of degree 2. $(E) : x^2 - 3x = 4$ is a polynomial equation of degree 2.

Definition 8. Let P be a polynomial. A number α is said to be a **root** of P if $P(\alpha) = 0$.

Proposition 7. Let P be a polynomial of degree $n \geqslant 1$. For any number α, $P(\alpha) = 0 \iff P(x) = (x - \alpha)Q(x)$, where Q is another polynomial (of degree $n - 1$). In particular, P has at most n distinct roots.

This proposition is very useful : if the factorization of a polynomial is known, its roots can be deduced ; conversely, to factorize a polynomial, one simply needs to find its roots.

Example 23. Solve $(x-3)(x^2+x)=0$, then solve $(x-3)(x^2+x)\geqslant 0$.

This example shows that to solve an inequality involving a polynomial, we transform the problem into one of determining the polynomial's sign, which is solved by finding its roots.

The degree 1 case

Definition 9. A polynomial of degree 1 is any function f that maps x to $ax+b$, where $a\neq 0$. A degree-one equation is an equation of the form $f(x)=0$.

Proposition 8. The equation $f(x)=0$ has a unique solution, $\dfrac{-b}{a}$; the sign of f is the same as that of $-a$ on $]-\infty,\dfrac{-b}{a}]$ and the same as that of a on $[\dfrac{-b}{a},+\infty[$.

Example 24. Solve $3x-1\leqslant 0$.

2nd degree.

Definition 10.

A quadratic trinomial is defined as any function f that maps x to ax^2+bx+c, where $a\neq 0$. The quantity $\Delta=b^2-4ac$ is called the discriminant of the equation.

Definition 11. Vertex form of a trinomial

$$ax^2+bx+c=a(x^2+\frac{b}{a}x+\frac{c}{a})=a\left[(x+\frac{b}{2a})^2-\frac{b^2}{4a^2}+\frac{c}{a}\right]=a\left[(x+\frac{b}{2a})^2-\frac{b^2-4ac}{4a^2}\right]$$

Example 25. Find the vertex form of x^2-2x+3.

Proposition 9.
To solve the equation $f(x)=0$ and factor the quadratic trinomial, we obtain the following three cases :

- if $\Delta>0$, f has two distinct roots :

  $$x_1=\frac{-b-\sqrt{\Delta}}{2a}\quad x_2=\frac{-b+\sqrt{\Delta}}{2a}$$

  yielding the factorization of f :

  $$f(x)=a(x-x_1)(x-x_2).$$

  We deduce that f has the same sign as a outside the roots and the same sign as $-a$ between the roots.
- if $\Delta=0$, there is a so-called double root

  $$x_0=\frac{-b}{2a}$$

  yielding the factorization of f

  $$f(x)=a(x-x_0)^2$$

  so f has the same sign as a.

- if $\Delta < 0$, f has no real roots ; f has the same sign as a on $\mathbb{R}$. However, f has two complex conjugate roots :

$$x_1 = \frac{-b - i\sqrt{|\Delta|}}{2a} \quad x_2 = \frac{-b + i\sqrt{|\Delta|}}{2a}$$

Proposition 10. Sum and product of roots
If $\Delta > 0$, then $f(x) = ax^2 + bx + c = a(x - x_1)(x - x_2)$, and

$$x_1 x_2 = \frac{c}{a} \quad x_1 + x_2 = \frac{-b}{a}$$

Remarque 2. You do not need to explicitly calculate Δ if the expression is a standard algebraic identity of the form $a^2 - b^2$ or $a^2 + 2ab + b^2$, or if one of the roots is obvious.

Example 26. Solve in $\mathbb{R}$

1. $x^2 - 4 = 0$
2. $x^2 - 5x = 0$
3. $x^2 + 1 = 0$
4. $x^2 - 2x + 5 = 2$
5. $x^2 + x + 1 \geqslant 0$
6. $x^2 - 4x - 77 \geqslant 0$
7. Find the pair of real numbers (x, y) such that $x + y = 8$ and $xy = 12$.

Higher-degree equations and inequalities

To solve an equation of the form $P(x) = 0$, where P is a polynomial of degree strictly greater than two, one looks for an obvious root of P—that is, a real number a such that $P(a) = 0$—and then factors P by $x - a$ using the Euclidean division algorithm.

Theorem 2. If P and Q are two polynomials, then there exists a unique pair (Q, R) such that $A = BQ + R$ and $\deg(R) < \deg(Q)$. Q is called the quotient, and R the remainder.

Example 27. Solve

1. $x^3 - 8 = 0$
2. $-4x^3 + 16x^2 - 4x - 24 = 0$
3. $x^3 - 2x^2 + x \leqslant 0.$

Rational equations

Definition 12. Let P and Q be two polynomial functions. Then $F(x) = \dfrac{P(x)}{Q(x)}$ is called a rational function (or, loosely speaking, a **rational fraction**). To solve an equation or inequality involving rational fractions, one attempts to simplify the fractions if possible and then transforms the problem into one involving polynomials.

Note that a fraction is defined only where its denominator is non-zero.

Example 28.

1. Solve $\dfrac{x^2 - x + 1}{x + 1} = 3.$
2. Simplify $\dfrac{x^2 - 2x + 1}{x - x^2}$ and solve $\dfrac{x^2 - 2x + 1}{x - x^2} \leqslant 3.$

5.2 Linear systems of two equations with two unknowns

Definition 13. A linear system of two equations with two unknowns is a system of the form

$$(S)\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$$

where a, b, c, d are rea numbers. Solving such a system means finding the set of pairs (x, y) that satisfy both equations of the system.

To solve such a system, we will transform it into an equivalent system using **linear combinations** (of the form $L_i \leftarrow L_i + aL_j$ for a suitably chosen coefficient a) until a conclusion can be reached.

Example 29. Solve

$$(S)\begin{cases} 2x+y \;\;=3 \\ -4x+5y=1 \end{cases}$$

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5.3 Equations and Inequalities Involving Absolute Values

Definition 14.
Let x be a real number. The absolute value of x is denoted by $|x|$, and we have :
$|x| = x$ if x is positive, $|x| = -x$ otherwise.

Example 30.
Write the following without absolute value signs : $\left|\dfrac{\sqrt{2}-1}{\sqrt{2}-3}\right|$.

Property 1.

- $|xy| = |x| * |y|$
- $|-x| = |x|$
- Triangle inequalities : $||a| - |b|| \leqslant |a+b| \leqslant |a| + |b|$
- $\sqrt{x^2} = |x|$

Example 31.
Give examples where the inequalities are strict, and where equality holds in the triangle inequalities.

The absolute value is often used to characterize real numbers belonging to an interval, thanks to the following property :

Property 2.
Let x and a be two real numbers and ε be a strictly positive real number.

- $|x-a| = \text{distance}(a; x)$
- $|x-a| \leqslant \varepsilon \Longleftrightarrow x \in [a-\varepsilon; a+\varepsilon]$
- $|X| \leqslant \varepsilon \Longleftrightarrow X \in [-\varepsilon; \varepsilon]$
- $|X| \geqslant \varepsilon \Longleftrightarrow X \in]-\infty; -\varepsilon] \cup [+\varepsilon, +\infty[$

Example 32. Solve

1. $|x-1| = -1$
2. $|2x+3| \geqslant 4$

6 Exercices

6.1 TD1

Exercise 1. Calculate z_1+z_2, z_1z_2, and $\frac{z_1}{z_2}$ for the complex numbers $z_1 = 2+i$ and $z_2 = 1+2i$.

Exercise 2.
Consider the proposition $P : x \geqslant 1$.

Q1 : $x^2 \geqslant 1$.

Q2 : $x^2 < 0$.

Q3 : $x - 2 = 0$.

Q4 : $x \leqslant 2$.

Q5 : $\exists y \geqslant 0$ such that $x = e^y$.

Q6 : $x \in [2, 3]$.

State whether each proposition is, with respect to P, a necessary but not sufficient condition, a sufficient but not necessary condition, a necessary and sufficient condition, or neither.

Exercise 3.

State whether the following propositions are true or false :

1. $\exists x \in \mathbb{R}, (x + 1 = 0 \text{ and } x + 2 = 0)$
2. $(\exists x \in \mathbb{R}, x + 1 = 0)$ and $(\exists x \in \mathbb{R}, x + 2 = 0)$
3. $\forall x \in \mathbb{R}, (x + 1 \neq 0 \text{ or } x + 2 \neq 0)$
4. $\forall \varepsilon > 0, \exists a \in \mathbb{R}, -\varepsilon \leqslant a \leqslant \varepsilon$
5. $\exists a \in \mathbb{R}, \forall \varepsilon > 0, -\varepsilon \leqslant a \leqslant \varepsilon$

Exercise 4.
Let f be a function of a real variable ; negate the following assertions.

1. $\forall x \in \mathbb{R}, \quad f(x) \neq 0.$
2. $\forall M > 0 \quad \exists A > 0, \forall x \geqslant A, f(x) > M.$
3. $\forall x \in \mathbb{R}, (f(x) > 0 \Longrightarrow x \leqslant 0).$
4. $\forall (a, b) \in \mathbb{R}^2 \,; a \leqslant b \implies f(a) \leqslant f(b).$

6.2 TD2

Exercise 5. let be $q \neq 1$ and $n \geqslant 1$. Prove that

$$\sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q}.$$

Exercise 6. Prove the irrationality of $\sqrt{2}$ by contradiction.

Exercise 7. Prove the following propositions using the contrapositive :

1. "If x is a real number such that $x^2 < 1$, then $x > -1$."
2. Let a be a real number ; show that : $\forall \varepsilon > 0, -\varepsilon \leqslant a \leqslant \varepsilon \implies a = 0$.

Exercise 8.
Expand the following expressions :

1. $(x-1)(x+2) - x(x-1)$
2. $(-3x-1)^2$
3. $(x-2)^3$
4. $(2x-1)^4$

Exercise 9.
Factorise the following expressions :

1. $2x^3 - 5x^2$
2. $x^2 - 1 + (3x-2)(x+1)$
3. $(x-1)(3x+2) + (5x-3)(-x+1)$
4. $2np - px + p$
5. $3a^2b + ab - 4ab^2$
6. $x^n + nx^{2n}$
7. $2^n + 8^n$

Exercise 10.

Determine the real numbers x such that $x = \sqrt{2x+3}$

6.3 Tutorial 3 and start of Tutorial 4

Exercise 11. Solve in $\mathbb{R}$:

1. $x^2 + 4x + 15 = 0$
2. $x^2 + 4x + 4 = 0$
3. $-2x^2 + 3x + 5 = 0$
4. $\dfrac{1}{6x} = \dfrac{1}{x^2+9}$
5. $\dfrac{x}{x^2-4} = \dfrac{1}{x+2}$
6. $\dfrac{1}{x} + \dfrac{1}{x+4} \leqslant 1$
7. without calculations $x^2 - (\sqrt{2}+\pi)x + \pi\sqrt{2} = 0$
8. $(3x+4)^2 < (x-3)^2$
9. $x^3 - 25x \geqslant 0$
10. $(x+1)^2 > (x+1)(1-2x)$
11. $x^3 - 5x^2 + 8x - 4 \geqslant 0$
12. $1 - 8x^3 = (2-4x)(x+2)$
13. Find the pair of real numbers (x, y) such that $x + y = -4$ and $xy = 3$.

Exercise 12. Solve in $\mathbb{C}$:

1. $x^2 + 4x + 5 = 0$.
2. $x^3 + 4x = 0$.
3. $x^2 + 4x - 3 = 0$.

6.4 Fin TD4 et TD5

Exercise 13.
Determine the real numbers x such that :

1. $|x-2|<3$
2. $|x+4|<-1$
3. $|x+5|\leqslant 3$
4. $|x+2|\geqslant 5$
5. $|x+5|\geqslant -1$
6. $|x-2|<5$ et $|x-3|\leqslant 4$
7. $|x+1|<1$ ou $|x+2|\leqslant 4$
8. $|x-5|<2$ et $|x+1|\leqslant 1$
9. $|x-2|<4$ ou $|x+2|\leqslant 2$

Exercise 14.
Determine the points $M(x;y)$ in the plane such that :

1. $|x+y|=2$
2. $|x+y|\leqslant 2$
3. $|x|+|y|\leqslant 3$

Exercise 15.

Solve the following systems

$$(S1)\begin{cases} x+y &= 7 \\ -5x+3y &= 2 \end{cases}$$

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$$(S2)\begin{cases} 2x-5y &= 3 \\ -4x+10y &= -6 \end{cases}$$

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$$(S3)\begin{cases} 3x-4y &= 6 \\ -2x+7y &= 1 \end{cases}$$

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