CURVILINEAR INTEGRAL

1 Curvilinear Integral

1.1 Introduction : work of a force

In what follows, it will be important to keep in mind that we shall be working on curves C that will be parametrized in terms of a single variable as follows,

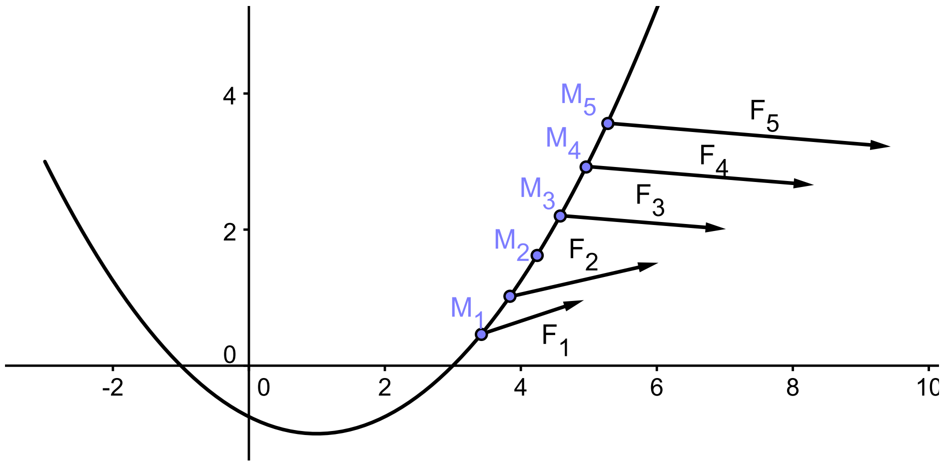
Example 1 (Examples of parametric curves).

1. Let's consider a parametric curve defined by $r(t) = (x(t), y(t))$ with $x(t) = 2t + 1$ and $y(t) = t^2$. Plot $M(t)$ for the parameters $0, 1, 2$.

2. Draw this parametric curve $x(t) = r\cos(t) \quad y(t) = r\sin(t)$

3. You get a cartesian equation : $y = f(x)$, give a parametric equation.

Let's consider a force $\vec{F}$ with coordinates (F_X, F_Y) in an orthonormal frame $(O, \vec{i}, \vec{j})$, applied to M. We assume that M browses along a parametric curve (C). Its position is given by its parametric coordinates $(x(t), y(t))$ for $t \in [a, b]$, the force $\vec{F}$ depends on the position of M, that is $\vec{F}(F_X(t), F_Y(t))$.



We approximate the work of $\vec{F}$ on (C) by $W = \sum \overrightarrow{M_iM_{i+1}}.\vec{F}$, with M_iM_{i+1} small thus we get $\overrightarrow{M_iM_{i+1}}$ has for coordinates (dx, dy). Thus the dot product $\overrightarrow{M_iM_{i+1}}.\vec{F_i} = F_{xi}dx + F_{yi}dy$ If the distance M_iM_{i+1} is close to 0, the sum goes to the integral : $W = \int F_X dx + F_Y dy = \int_a^b \vec{F}.\vec{dl}$ with $\vec{dl}(dx(t), dy(t))$ that is $\vec{dl}(x'(t)dt, y'(t)dt)$. Note that coordinates of (F_X, F_Y) depends on x and y so we could write: $W = \int f(x,y)dx + g(x,y)dy$

Thus we have $W = \int_a^b F_X(t)x'(t) + F_Y(t)y'(t)dt$

This integral is called the *circulation* $\vec{F}$ along (C).

Example 2.
Compute the work of the weight for a mobile (its mass is m) moving on $[AB]$ from $A(0,1)$ to $B(1,0)$.

1.2 To integrate a differential form

1.2.1 General case

Definition 1.
In this section, we will work with a differential form ω from Ω to $\mathbb{R}$ defined by $\omega(x,y) = P(x,y)dx + Q(x,y)dy$, where P et Q are two functions of differentiability class C^1 defined from Ω to $\mathbb{R}$ where Ω is an open set of $\mathbb{R}^2$.
To simplify, ω is defined on $\mathbb{R}^2$, but all the formula could be generalized for a linear form $\mathbb{R}^p$ to $\mathbb{R}$.

Example 3.
Let $P(x,y) = 2xy$ and $Q(x,y) = x^2$, write the associated differential form.

Definition 2.
Let (C) be a parametric curve $(x(t), y(t))$ with $t \in [a,b]$.
Then the curvilinear integral ω along (C) is defined by :

$$\oint_{(C)} P(x,y)dx + Q(x,y)dy = \int_a^b P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)dt$$

Example 4.

Let $\omega = xydx + xdy$
Calculate $\oint_{(C)} \omega$, where (C) has for equation :
$y = \sqrt{x}$, x goes from 1 to 2.

Remarque 1.

The parameterization of (C) implies an orientation of the curve, and in particular we have:

Proposition 1.
Let (C^+) be the curve (C) with a direction and (C^-) be the curve in the other direction. Thus we get : $\oint_{(C^+)} \omega = -\oint_{(C^-)} \omega$

Proposition 2.
The integral of a function does not depend on the parametric representation of the curve, the curve is described in the same direction.

Example 5.
Let $\omega = xy^2dx - x^2ydy$
Let (C) be the circle with those parametric representations $\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases}$ and $\begin{cases} x(t) = \sin(t) \\ y(t) = \cos(t) \end{cases}$
Calculate $\oint_{(C)} \omega$ using the two parametrizations, we use the trigonometrical direction for the curve.

1.2.2 Exact differential forms

Definition 3.
ω is an exact differential form if there exists a function u such that $\omega = du$, where $\omega = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$.

Proposition 3.
We assume that Ω is simply connected open space of $\mathbb{R}^2$ (informally, an object in our space is simply connected if it consists of one piece and does not have any holes that pass all the way through it. For example, neither a doughnut nor a coffee cup with a handle is simply connected,), then the differential form ω is exact if and only if:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Example 6. Let ω be defined by $\omega = xy^2dx - x^2ydy$. Is ω an exat differential form ?

Proposition 4.
If ω is an exact differential form ($\omega = du$), and $(C) = \widehat{AB}$ then :

- The integral of ω along a closed curve is zero
- The integral of ω along a curve only depends on the first and the last points so $\oint_{(C)} \omega = u(B) - u(A)$

Example 7. Prove the previous property.

Example 8. Let ω_1 and ω_2 defined by $\omega_1 = xy^2dx - x^2ydy$ and $\omega_2 = xdx + ydy$. Calculate the curvilinear curve for those forms on the circle of center 0 et radius 1.

1.3 Green-Riemann Formula or theorem

1.3.1 Formula

Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If $P = P(x, y)$ and $Q = Q(x, y)$ are functions of (x, y) defined on an open region containing D and having continuous partial derivatives there, then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_{(C)} Pdx + Qdy$$

where the path of integration along C is anticlockwise.

Remarque 2.
This allows to compute a double integral using a curvilinear integral. . The choice for P and Q is not unique. We choose P and Q to simplify our computations. This formula is also used to compute a curvilinear integral using a double integral.

Example 9. Calculate $I = \iint_D y dxdy$ where $D = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leqslant 1\}$ directly and using the above formula.

1.3.2 Area calculus

Proposition 5.
Let D be a compact of frontier (C). Then the area A of K is equal to :

$$A = \iint_D dxdy = \frac{1}{2}\oint_{(C)} xdy - ydx = \oint_{(C)} xdy = \oint_{(C)} -ydx$$

2 Exercices

Exercice 1.
Let's consider the differential form $\omega = 2xe^y dx + x^2 e^y dy$, defined on $\mathbb{R}^2$.
Prove that ω is exact. Find antiderivatives on $\mathbb{R}^2$..

Exercice 2.
Let's consider the differential form $\omega = \dfrac{2x}{y}dx - \dfrac{x^2}{y^2}dy$, defined on the half plane $U = \{(x, y) \in \mathbb{R}^2; y > 0\}$.

1. Prove that ω is exact.
2. Calculate $\displaystyle\oint_{(C)} \omega$ where (C) a piecewise C^1 curve with the starting point $A(1, 2)$ and last point $B(3, 8)$.

Exercice 3.
Let's consider the differential form $\omega = (x + y)dx + (x - y)dy$. defined on the half-plane $U = \{(x, y) \in \mathbb{R}^2; y > 0\}$.

1. Prove that ω is exact.
2. Calculate $\displaystyle\oint_{(C)} \omega$ where (C) a piecewise C^1 curve with the starting point $A(1, 2)$ and the last point $B(3, 8)$.

Exercice 4.
Calculate the curvilinear integral $\displaystyle\oint_{(C)} \omega$ in the following examples :

1. $\omega = xy\,dx + (x + y)dy$ where (C) is the arc of the parabola $y = x^2, -1 \leqslant x \leqslant 2$, we use the clockwise direction.
2. $\omega = y \sin x dx + x \cos y dy$ where (C) is the part of the line $[OA]$ from $O(0, 0)$ to $A(1, 1)$.

Exercice 5.
Calculate the curvilinear integral $\omega = x^2 dx - xy\,dy$ along :

1. the part of the line $[OB]$ from $O(0, 0)$ to $B(1, 1)$.
2. the arc of the parabola $x = y^2, 0 \leqslant x \leqslant y$, we use the clockwise direction.

Exercice 6.
Calculate the curvilinear integral $\omega = \dfrac{x - y}{x^2 + y^2}dx + \dfrac{x + y}{x^2 + y^2}dy$ along the square ABCD, avec A(1,1), B(-1,1), C(-1,-1) and D(1,-1), we use the clockwise direction.

Exercice 7. Compute Green-Riemann's formula

1. $\displaystyle\iint_D y\,dxdy$ where $D = \{(x, y) \in \mathbb{R}^2/(x - 1)^2 + y^2 \leqslant 1\ ; y \geqslant 0\}$,
2. Let $0 < b < a$. $\displaystyle\iint_D \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) dxdy$ where $D = \{(x, y) \in \mathbb{R}^2/\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leqslant 1\}$