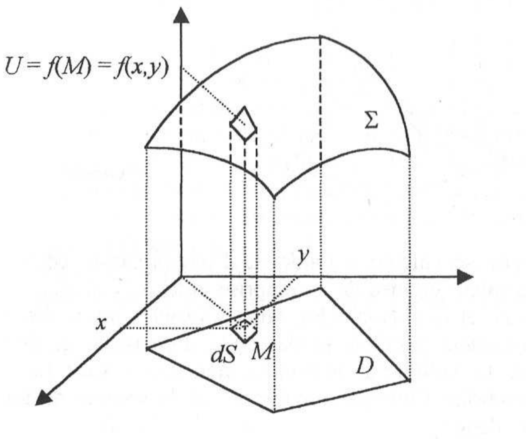
DOUBLE INTEGRALS

The definite integral can be extended to functions of more than one variable. This chapter is a generalization for functions of two variables. Usual properties of definite integrals are true. We will focus on how to compute a double integral.

1 Generalities

Let f be a function of two variables x et y defined on a set $D \subset \mathbb{R}^2$.
For each point $M(x, y)$ of D we draw $U = f(M)$, by the way we get the graph of f, this is a surface denoted by Σ.



Let's consider an infinitesimal surface dS around M. $f(M)dS$ deals with the volume of the infinitesimal prism drawn previously. This prism has a basis dS and its height is $U = f(M)$. This volume is positive is U is above the xy plane and negative if not.
If we sum all the volumns of thoses prisms $f(M)dS$ for all the points $M \in D$, we get a double integral :

$$I = \int\int_D f(M)dS$$

Mathematically, a double integral represents the algebraic volume between the xy plane delimited by D and the surface Σ. The double integral is equal to the algebraic volume under the surface z=f(x,y) and above xy-plane for x and y in the region D. The notation $\int\int$ is due to the fact that the area of integration is a surface. We are going to compute two iterated integrals.

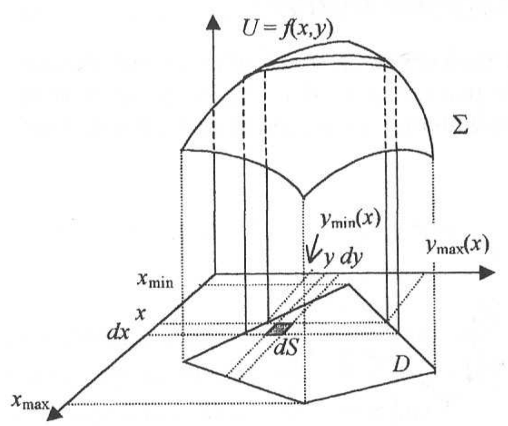
2 With Cartesian Coordinates

Using cartesian coordinates, we get the surface dS as x varies of dx and y of dy. The length and width of the rectangle are dx and dy, respectively. Hence dydx (or dxdy) is the area of the rectangle. Thus $dS = dxdy$.

So we have

$$I = \int\int_D f(x,y)dxdy$$

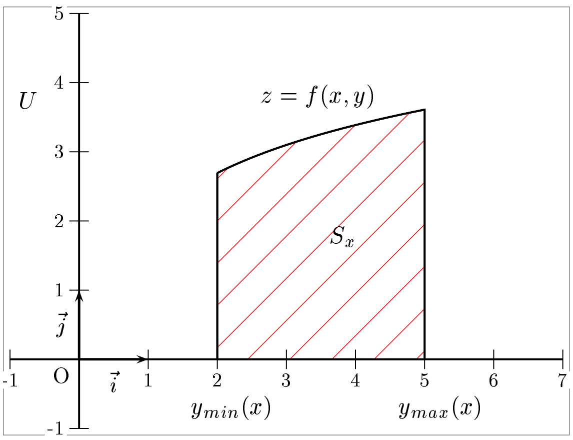
For a **given** value x between x_{min} and x_{max}, wa assume that x varies of dx. We can compute the volume by slicing the three-dimensional region like a loaf of bread. Suppose the slices are parallel to the y-axis. An example of slice between x and x+dx is shown in the figure.
In the limit of infinitesimal thickness dx, the volume of the slice is the product of the cross-sectional area and the thickness dx. The cross sectional area is the area under the curve f(x,y) for fixed x and y varying between two values.
Let's denote S_x the surface of this slice, it follows that the volume of the infinitesimal slice $dV_x = S_x dx$.



The total volume is the sum of the volumes of all the slices between $x = x_{min}$ and $x = x_{max}$:

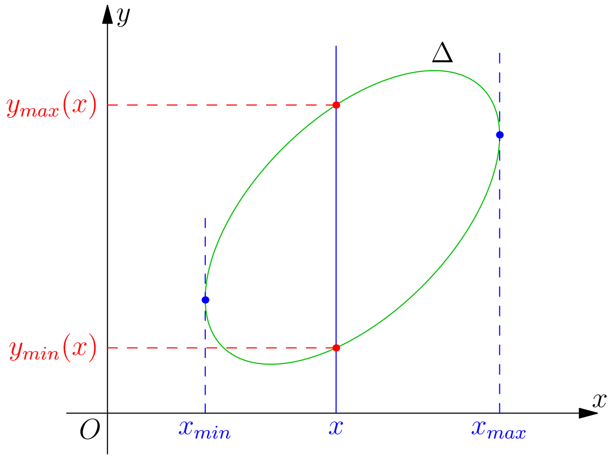
$$I = \int\int_D f(x,y)dxdy = \int_{x_{min}}^{x_{max}} dV_x = \int_{x_{min}}^{x_{max}} S_x dx$$

Therefore we have to find the expression of the cross-sectional area S_x.
S_x is the surface under the curve $f(x,y) = z$ for **a fixed** x (only y varies) and between vertical straight lines of equations $y = y_{min}(x)$ and $y = y_{max}(x)$.



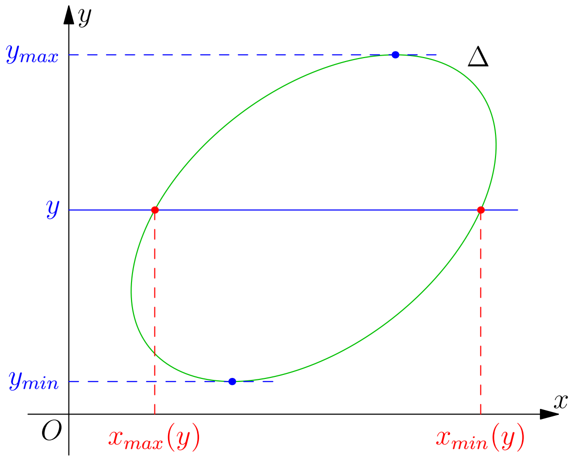
So we have $S_x = \int_{y=y_{min}(x)}^{y=y_{max}(x)} f(x,y)dy$. Therefore we get the fundamental formulae :

$$I = \int\int_D f(x,y)dxdy = \int_{x_{min}}^{x_{max}} \left(\int_{y=y_{min}(x)}^{y=y_{max}(x)} f(x,y)dy \right) dx$$



Alternatively, one can make slices that are parallel to the x-axis. In this case the volume is given by :

$$I = \int\int_D f(x,y)dxdy = \int_{y_{min}}^{y_{max}} \left(\int_{x=x_{min}(y)}^{x=x_{max}(y)} f(x,y)dx \right) dy$$

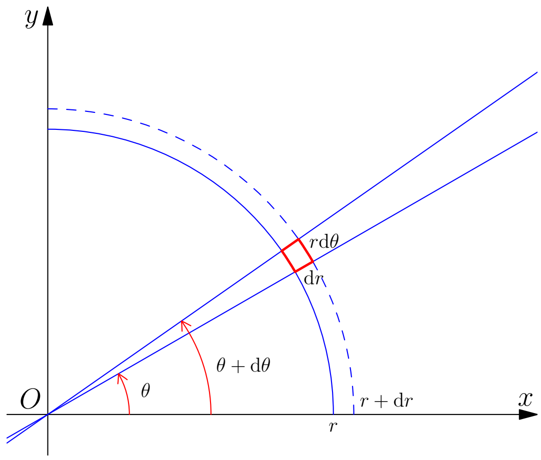


Example 1.
Calculate $\int\int_D xy dxdy$ where $D = \{(x,y) \in \mathbb{R}^2 / 2x + y \leqslant 2; x + y \geqslant 1; x \geqslant 0\}$
Before starting, it is useful to draw the region area in order to choose the formula to apply. Of course whatever is the formula choosen, you should get the same result.

3 With Polar coordinates

In this section we want to look at some regions that are much easier to describe in terms of polar coordinates. For instance, we might have a region that is a disk, ring, or a portion of

a disk or ring. In these cases using Cartesian coordinates could be somewhat cumbersome. The problem is that we can't just convert the dx and the dy into a dr and θ .
dS is an angular sector, we do as if it was an infinitesimal rectangle of length dr and of width $rd\theta$. Thus $dS = rdrd\theta$.



Finally we will compute I using two iterated inegrals :

$$I = \int\int_D g(r,\theta) r dr d\theta = \int_{\theta=\theta_{min}}^{\theta=\theta_{max}} \left(\int_{r=r_{min}(\theta)}^{r=r_{max}(\theta)} r g(r,\theta) dr \right) d\theta$$

Example 2.
Compute using polar coordinates $I = \int\int_D \frac{1}{x^2+y^2} dxdy$ where
$D = \{(x,y) \in \mathbb{R}^2 / 1 \leqslant x^2 + y^2 \leqslant 4; x \geqslant 0; y \geqslant 0\}$

4 Double integrals over rectangular regions

Let $I = \int\int_D f(u,v) du dv$ be a double integral. D is a rectangular region if $v_{min}(u)$ and $v_{max}(u)$ are **independent** of u.
In this particular case, we get :

$$I = \int\int_D f(u,v) du dv = \int_{u=u_{min}}^{u=u_{max}} \int_{v=v_{min}}^{v=v_{max}} f(u,v) du dv$$

With cartesian coordinates, a rectangular region is a rectangle whose sides are parallel to the axis. With polar coordinates, a rectangular region is a circular sector.
If moreover on this rectangular region D, we have $f(u,v) = g(u)h(v)$, f is said of separated variables and applying Fubini's theorem we get :

Theorem 1.

$$I = \int\int_D f(u,v) du dv = \int_{u=u_{min}}^{u=u_{max}} g(u) du \times \int_{v=v_{min}}^{v=v_{max}} h(v) dv$$

Thus we compute a double integral as a product of two definite integrals.

Example 3.
Calculate $\int\int_D xy\,dxdy$ where D is the rectangle $[a;b] \times [c;d]$.

5 To compute the area of the region D

We know that $I = \int\int_D f(u,v)dudv$ is linked to the volume under Σ and above the region D.
It is possible to use a double integral $I = \int\int_D$ so as to compute the area of the region D. It sufficies to take $f(x,y) = Constant$ in particular we take $f(x,y) = 1$. So, the area A of the region D is :

$$A = \int\int_D 1dudv$$

and so with cartesian coordinates

$$A = \int\int_D dxdy$$

and so with polar coordinates

$$A = \int\int_D rdrd\theta$$

.

Example 4.
Compute the area A of the region in the xy plane xOy bounded by $2y = 16 - x^2$ and $x + 2y = 4$

6 Change of variables : general case

The goal of this section is to compute $\int\int_D f(x,y)dxdy$ using a change of variables.
Let ϕ be a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. We assume that

- ϕ is a bijection from the interior of D to R
- ϕ is differentiable
- its inverse function is differentiable

We get two possibilities (as we had for simple integrals) :

6.1 u et v are given in function of x and y : $(u,v) = \phi(x,y)$

The Jacobian of (u,v) denoted by $J(x,y)$ is defined as follows $J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial x}$

We assume that the jacobian of (u, v) does not vanish on D
Thus we have :

$$|J(x, y)|dxdy = dudv$$

.
We get a function g such that $f(x, y) = g(u(x, y), v(x, y))|J(x, y)|$.
And : $\int\int_D f(x, y)dxdy = \int\int_R g(u, v)dudv$
with $R = \{(u, v) \in \mathbb{R}^2 / u = u(x, y), v = v(x, y)$ avec $(x, y) \in D\} = \phi(D)$

Example 5.
Calculate $\iint_D (x + 2y)(2x + y)^2 dxdy$ where $D = \{(x, y) \in \mathbb{R}^2$ st $1 \leqslant x + 2y \leqslant 2$ and $1 \leqslant 2x + y \leqslant x + 2y$ by setting $u = x + 2y$, $v = 2x + y$

6.2 x and y are given in function of u and v : $(x, y) = \phi^{-1}(u, v) = \varphi(u, v)$

The Jacobian of (x, y) is denoted and defined by :

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

We get $\varphi(R) = D$, thus :

$$\int\int_D f(x, y)dxdy = \int\int_R f(\varphi(u, v)) |J((u, v)| dudv$$

Example 6.
Find again the formula for polar coordinates.

7 In physics

If we know $\sigma(x, y)$ the surface density of a plate Δ then its mass is given by the formula :

$$M = \int\int_\Delta \sigma(x, y)dxdy$$

and its center of mass $G = (x_G, y_G)$ is such that :

$$\overrightarrow{OG} = \frac{1}{M}\int\int_\Delta \overrightarrow{OP}\sigma(x, y)dxdy$$

with the vector $\overrightarrow{OP} = (x, y)$, which means that we have :

$$x_G = \frac{1}{M}\int\int_\Delta x\sigma(x, y)dxdy$$
$$y_G = \frac{1}{M}\int\int_\Delta y\sigma(x, y)dxdy$$

8 Exercises

Exercise 1.
Express the double integral $\iint_D f(x,y)dxdy$ on the following regions :

1. D : rectangular region of vertices $(-1,-1),(2,-1),(2,4)$ et $(-1,4)$.
2. D : triangular region of vertices $(2,9),(2,1),(-2,1)$. (At home)
3. D : region bounded by : $y=\sin x, y=\cos x, x=0, x=\frac{\pi}{4}$
4. D : region bounded by : $y=x^2, y=0, x=2$
5. D :region bounded by : $y=2x, y=-x, y=4$ (At home)
6. D : region bounded by : $x=2\sqrt{y}, \sqrt{3}x=\sqrt{y}, y=2x+5$

Exercise 2.
Compute the following double integrals on the given regions :

1. $\iint_D x^2y\,dxdy$ where $D=\{(x,y)\in\mathbb{R}^2/x\geqslant 0; y\geqslant 0; x+y\leqslant 1\}$
2. $\iint_D \frac{1}{(x+y)^2}dxdy$ where $D=\{(x,y)\in\mathbb{R}^2/x\geqslant 1; y\geqslant 1; x+y\leqslant 4\}$
3. $\iint_D \sqrt{x}\,dxdy$ where $D=\{(x,y)\in\mathbb{R}^2/x\geqslant 0; y\geqslant 0; x^2\leqslant y\leqslant x\}$
4. $\iint_D \frac{1}{(1+x^2)(1+y^2)}dxdy$ where $D=\{(x,y)\in\mathbb{R}^2/0\leqslant y\leqslant x\leqslant 1\}$

Exercise 3.
Compute the following double integrals on the given regions :

1. $\iint_D \frac{1}{1+x^2+y^2}dxdy$ where $D=\{(x,y)\in\mathbb{R}^2/x^2+y^2\leqslant 1\}$
2. $\iint_D (x^2-y^2)dxdy$ where $D=\{(x,y)\in\mathbb{R}^2/0\leqslant y\leqslant x; x^2+y^2\leqslant R^2\}$
3. $\iint_D y\,dxdy$ where $D=\{(x,y)\in\mathbb{R}^2/(x-1)^2+y^2\leqslant 1\,; y\geqslant 0\}$, using cartesian coordiantes at first and then the change of variables $x=1+r\cos\theta$ and $y=r\sin\theta$.
4. $\iint_D (x^2+y^2)dxdy$ where $D=\{(x,y)\in\mathbb{R}^2/x^2+y^2-2y\leqslant 0\}$
5. $\iint_D e^{-(x^2+xy+y^2)}dxdy$ where $D=\{(x,y)\in\mathbb{R}^2/x^2+xy+y^2\leqslant 1\}$ We will use the change of variables $u=x+0.5y$ et $v=\frac{\sqrt{3}}{2}y$.

6. $\displaystyle\iint_D xy\,dx\,dy$ where $D = \{(x,y) \in \mathbb{R}^2 / x \geqslant 0, y \geqslant 0, x^{\frac{2}{3}} + y^{\frac{2}{3}} \leqslant 1\}$

   We will use the change of variables $x = r\cos^3\theta$ and $y = r\sin^3\theta$.

Exercise 4. (Optional)

1. Let $a > 0$. Can you compute $\displaystyle\int_0^a e^{-t^2}dt$?
2. Calculate $\displaystyle\iint_{D_a} e^{-(x^2+y^2)}dx\,dy$ where D_a is the disk of center O and of radius a.
3. Let's denote C_a the square of center O and of side $2a$ and $D_{\sqrt{2}a}$ the disk of center O and of radius $\sqrt{2}a$. Find a double inequality between $\displaystyle\iint_{D_a} e^{-(x^2+y^2)}dx\,dy$, $\displaystyle\iint_{C_a} e^{-(x^2+y^2)}dx\,dy$ and $\displaystyle\iint_{D_{\sqrt{2}a}} e^{-(x^2+y^2)}dx\,dy$.
4. Using Fubini's theorem and the sandwich theorem, let's deduce the limitof $\displaystyle\int_0^a e^{-t^2}dt$ as $a \to +\infty$.