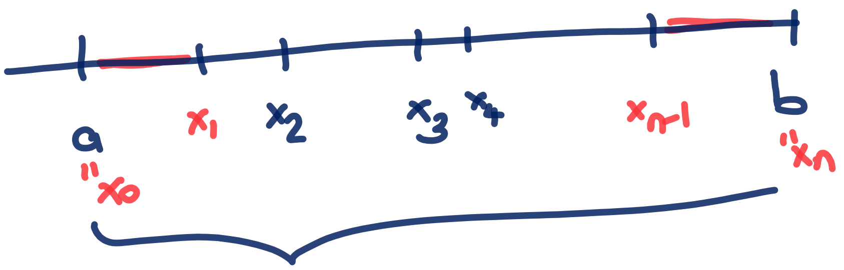
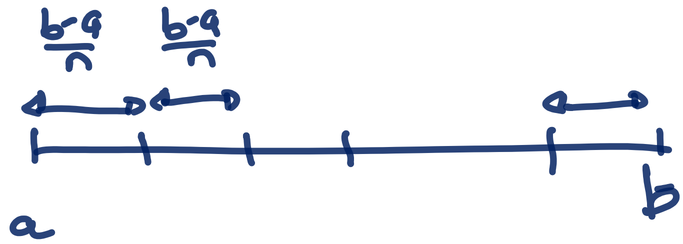
- 

n pieces $\Rightarrow$ $(n+1)$ points

- 

Regular subdivision

- Ex 2:

- On $[x_0,x_1]=[0,1]$: $f(x_1)=1$
 so the area of the first rectangle
 is $f(x_1)\,(x_1-x_0) = 1\times 1$

- On $[1,2]$: $f(x_2) = 6\ 5$
 So the area of the second rectangle
 is $f(x_2)\,(x_2-x_1) = 6.5\times 1$

- On $[2,3]$: $f(x_3)=8$
 area : 8×1

- On $[3,4]$: $f(x_4)=7$ area: 7

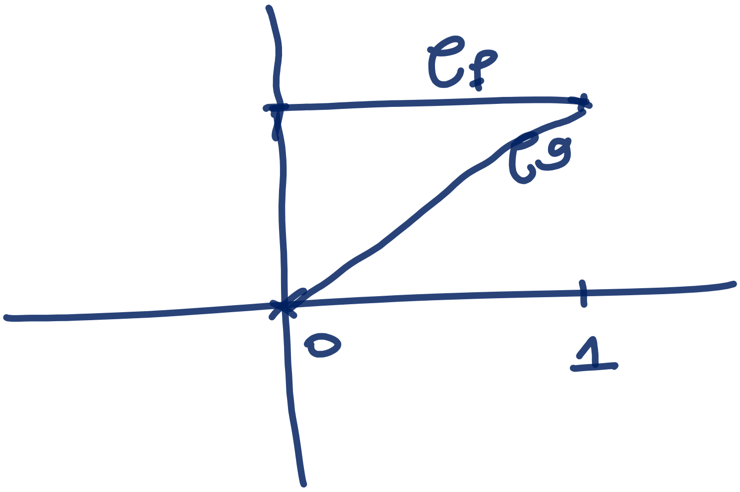
- On $[4,5]$: $f(x_5)=6.5$, area $\cdot$ 6.5

- On $[5,6]$: $f(x_6) = 8$ area 8×1

Finally

$$S(f,\sigma) = 1 + 6.5 + 8 + 7 + 6.5 + 8$$
$$= 37$$

Ex 3:



- $[0,1]$: $x_0 = 0$, $x_n = 1$ and $x_i = i/n$ $0 \leq i \leq n$

- Area under f: area of a square $A_{C_f} = 1 \times 1 = 1$
- $A_{C_g} = 1/2 \times 1 = 1/2$

- Rectangle method for f:

$$S(f,\sigma) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} 1 \times \frac{1}{n} = 1$$

, as $x_i = \frac{i}{n}$ $x_{i-1} = \frac{i-1}{n}$

$$S(g,\sigma) = \sum_{i=1}^{n} g(x_i)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} x_i \cdot \frac{1}{n}$$

$$= \sum_{i=1}^{n} \frac{i}{n} \times \frac{1}{n}$$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \xrightarrow[n \to \infty]{} 1/2$$

- <u>Linearity</u> :

$$* \int_a^b f+g = \lim_{n \to +\infty} \sum_{k=1}^{n} (f+g)\left(a + \frac{b-a}{n} \cdot k\right) \cdot \frac{b-a}{n}$$

$$= \lim_{n \to +\infty} \left[\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \cdot \frac{b-a}{n} + \sum_{k=1}^{n} g\left(a + k\frac{b-a}{n}\right) \frac{b-a}{n} \right]$$

, as both f and g are Riemann integrable we get

$$\int_a^b (f+g) = \lim_{n\to+\infty} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\frac{b-a}{n}$$

$$+ \lim_{n\to+\infty} \sum_{k=1}^{n} g\left(a+k\frac{b-a}{n}\right)\frac{b-a}{n}$$

$$= \int_a^b f + \int_a^b g$$

$$* \int_a^b \lambda f = \lim_{n\infty}\left(\sum_{k=1}^{n} (\lambda f)\left(a+k\frac{b-a}{n}\right) \cdot \frac{b-a}{n}\right)$$

$$= \lim_{n\infty}\left(\sum_{k=1}^{n} \lambda f\left(a+k\frac{b-a}{n}\right)\frac{b-a}{n}\right)$$

$$= \lim_{n\infty}\left(\lambda \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\frac{b-a}{n}\right)$$

$$= \lambda \lim_{n\infty}\left(\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)\frac{b-a}{n}\right)$$

$$= \lambda \int_a^b f.$$

Positivity: If $f \geq 0$ on $[a,b]$

then $f(x_i) \geq 0$

so

$$\lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{f\left(a+k\frac{b-a}{n}\right)}_{\geq 0} \underbrace{\frac{b-a}{n}}_{\geq 0} \geq 0$$

thus $\int_a^b f \geq 0$

Monotonicity:

If $g \geq f$ then $g-f \geq 0$

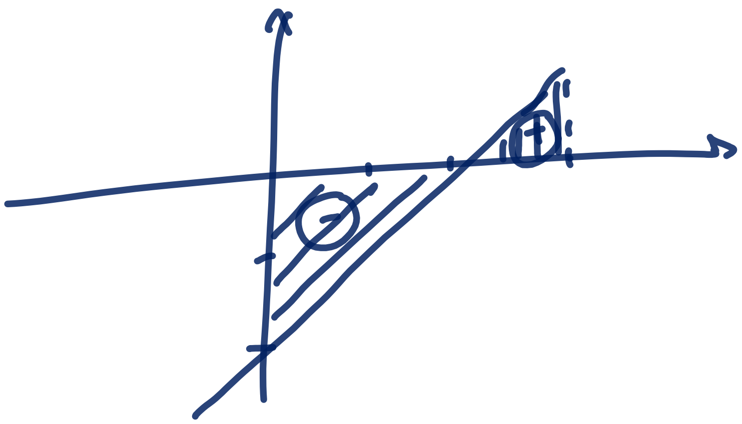
so by positivity $\int_a^b (g-f) \geq 0$

and by linearity $\int_a^b g - \int_a^b f \geq 0$

thus $\int_a^b g \geq \int_a^b f$.

Area Calculus

$$\int_0^3 (x-2)\,dx = \left[\frac{x^2}{2} - 2x\right]_0^3 = \frac{9}{2} - 6 = -3/2$$



Ex 4:

- One antiderivative is $x \longmapsto x^4$, as its derivative is $4x^3$

- All antiderivatives for $x \mapsto x^3$ is $x \longmapsto \frac{x^4}{4} + C \quad C \in \mathbb{R}$

$$\int x^3 \, dx = \frac{x^4}{4} + C$$

Ex 5:

$$\int_1^3 x^3 \, dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{84}{4} - \frac{1}{4} = \frac{83}{4}$$

There is no point in putting C, as we get endpoints :

$$= \left[\frac{x^4}{4} + C\right]_1^3 = \frac{84}{4} + C - \left(\frac{1}{4} + C\right)$$

$$= \underbrace{83/4}_{\text{we get the same result}}$$

Ex 6 :

1) $\int 2x\, e^{x^2} dx = \int u' e^{u} dx \qquad u(x) = x^2$

$= e^{x^2} + C$

2) $\int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\int \frac{u'(x)}{u(x)} dx$

$= -\ln|u(x)| + C \qquad C \in \mathbb{R}$

$= -\ln|\cos x| + C$

3) $\int \frac{2x}{x^2+1} dx = \int \frac{u'(x)}{u(x)} dx = \ln|x^2+1| + C$

$= \ln(x^2+1) + C$

4) $\int \frac{2}{4x^2+1} dx = \int \frac{2}{(2x)^2+1} dx$

$= \int \frac{u'(x)}{1+(u(x))^2} dx$

$= \text{Arctan}(2x) + C \quad C \in \mathbb{R}$

5) $\displaystyle\int \frac{1}{x} \ln x \, dx = \int u'(x)\, u(x)\, dx$

$= \frac{1}{2}(\ln x)^2 + C$

6) $\displaystyle\int_0^{\pi/2} \frac{1}{t+1} + \cos(t) + e^t \, dt$

$= \left[\ln|t+1| + \sin t + e^t \right]_{t=0}^{t=\pi/2}$

$= \ln\left(1+\frac{\pi}{2}\right) + 1 + e^{\pi/2} - (0 + 0 + 1)$

$= \ln(1+\pi/2) + e^{\pi/2}$

7) $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{1+(\cos x)^2} \, dx$

$u(x) = \cos x$

$u'(x) = -\sin x$

$$= -\int_0^{\pi/2} \frac{u'(x)}{1+(u(x))^2}\,dx$$

$$= -\Big[\operatorname{Arctan}(\cos x)\Big]_0^{\pi/2}$$

$$= -\left[\operatorname{Arctan}(0) - \operatorname{Arctan}(1)\right]$$

$$= 0 + \pi/4$$

Ex 7 :

$$F(x) = \int_{u(x)}^{v(x)} f(t)\,dt = \int_a^{v(x)} f(t)\,dt - \int_a^{u(x)} f(t)\,dt$$

$$= G(v(x)) - G(u(x))$$

with $G(X) = \int_a^X f(t)\,dt \Rightarrow G'(X) = f(X)$

so

$$F'(x) = v'(x)\,G'(v(x)) - u'(x)\,G'(u(x))$$

$$= v'(x)\,f(v(x)) - u'(x)\,f(u(x))$$

Ex 8 :

$$\int_0^1 x \sin(2x)\,dx = I$$

$u(x) = x$ $\qquad$ $v(x) = -\frac{1}{2}\cos(2x)$

$u'(x) = 1$ $\qquad$ $v'(x) = \sin(2x)$

$$I = \left[-\frac{1}{2}x\cos(2x)\right]_0^1 - \int_0^1 -\frac{1}{2}\cos(2x)\,dx$$

$$I = -\frac{1}{2}\cos(2) + 0 + \frac{1}{2}\int_0^1 \cos(2x)\,dx$$

$$I = -\frac{1}{2}\cos(2) + \frac{1}{2}\left[\frac{\sin(2x)}{2}\right]_0^1$$

$$I = -\frac{1}{2}\cos(2) + \frac{1}{4}\sin 2 - 0$$