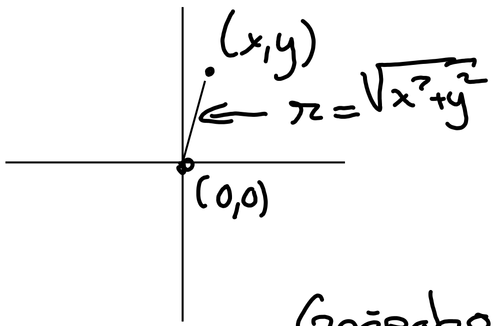
How to compute limits in $\mathbb{R}^2$.

- We say that $(x,y) \to (0,0)$ when $d((x,y),(0,0)) \to 0$

i.e. $\sqrt{x^2+y^2} \to 0$. This implies that $x \to 0$ & $y \to 0$.



- $f(x,y) \xrightarrow[(x,y)\to(0,0)]{} \ell$ means that $f(x,y)$ converges to ℓ no matter how the point (x,y) approaches $(0,0)$.

- Operations on limits work (composition, quotient, etc)

ex: - $3x - y^2 \xrightarrow[(x,y)\to(0,0)]{} 0$; $\exp(3x-y^2) \xrightarrow[(x,y)\to(0,0)]{} 1$

Pb: When there is an indetermination:

ex. $\dfrac{x}{x^2+y^2}$; $\dfrac{x^2}{x^2+y^2}$; $\dfrac{x^3}{x^2+y^2}$; $\dfrac{\sin(xy)}{xy}$, ...

↳ If we want to show that $f(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0$

- We try to find an upper bound that depends only on that is not indeterminated: $\sqrt{x^2+y^2} = r$, or

ex: - $\left|\dfrac{x^3}{x^2+y^2}\right| = \left|\dfrac{x^2 \cdot x}{x^2+y^2}\right| \leq |x|$ and $|x| \xrightarrow[(x,y)\to(0,0)]{} 0$

$\uparrow$ $x^2 \leq x^2+y^2$

So $\dfrac{x^3}{x^2+y^2} \xrightarrow[(x,y)\to(0,0)]{} 0$

- Or $f(x,y) = \dfrac{x^4+y^3}{x^2+y^2}$

Polar coordinates

$|f(r\cos\theta, r\sin\theta)| = \left|\dfrac{r^4\cos^4\theta + r^3\sin^3\theta}{r^2}\right|$

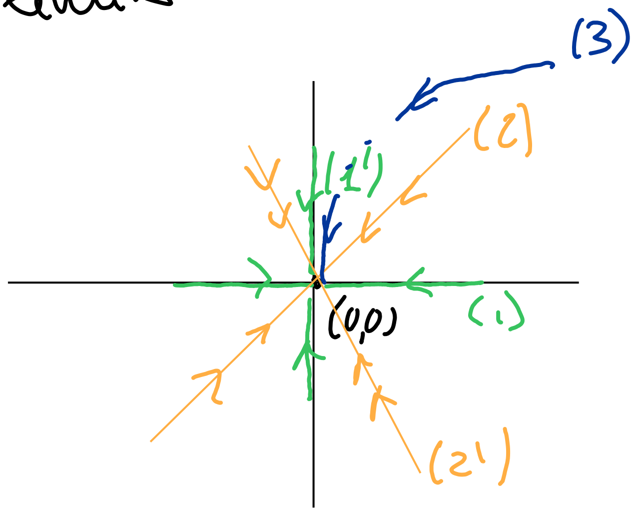
$\leq |r^2\cos^4\theta| + |r\sin^3\theta| \leq \underbrace{r^2 + r}_{\xrightarrow[r\to 0]{} 0}$

So $f(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0$

To show that there is no limit at $(0,0)$

- It is enough to exhibit two paths converging to $(0,0)$ but that give different limits.

<u>ex of usual paths:</u>



$(x_t, y_t) =$
- $(t,0)$ (1) } from 1 axis
- $(0,t)$ (1')
- (t,t) (2) } from one specific angle
  and $(t, \alpha t)$ (2')
- (t, t^α) $\alpha \in \mathbb{R}$, $\alpha > 0$ (3)
- ...

<u>ex:</u> $f = \dfrac{xy}{x^2+y^2}$:
- $f(t,0) = 0 \ \forall t \ : \xrightarrow[t \to 0]{} 0$
- $f(t,t) = \dfrac{1}{2} \ \forall t \ \xrightarrow[t \to 0]{} \dfrac{1}{2}$

} : No limit at $(0,0)$

- $f(x,y) = \dfrac{x}{y}$: $f(t, \alpha t) = \dfrac{t}{\alpha t} = \dfrac{1}{\alpha}$: will give $\neq$ limits for $\neq \alpha$!

↳ It is also possible to use polar coordinates: If we obtain $\neq$ behaviours when $r \to 0$ depending on what θ does:

<u>ex:</u> $\dfrac{xy}{x^2+y^2} = \dfrac{r^2 \cos\theta \sin\theta}{r^2} = \cos\theta \sin\theta$: Different limits when $r \to 0$ depending on the value of $\cos\theta\sin\theta$.

<u>ex:</u> $\dfrac{y + (x^2+y^2)^2}{(x^2+y^2)^2} = \dfrac{r\sin\theta + r^4}{r^4} = 1 + \dfrac{\sin\theta}{r^3}$: limit will depend on what θ does ($\sin\theta = 0 \Rightarrow \lim = 1$, $\sin\theta$ fixed $\neq 0 \Rightarrow \lim \infty$)