COMPLEX NUMBERS

Learning objectives

- **To know the rectangular and the polar form of a complex number.**
- **To be able to solve a complex equation.**
- **To become familiar with linearizing a sine and a cosine.**

Definition 1. Let's denote $\mathbb{C}$ the set of all complex numbers. The construction of the field of complex numbers is quite technical. A purely imaginary unit is defined, denoted by i such that :

$$i^2 = -1$$

The letter i refers to imaginary.
In electricity, complex numbers are very useful but the letter i refers to the intensity that is the reason why we use the letter j to denote a complex number.

There exists many ways to write a compex number z, depending on the framework.

1 Rectangular form

1.1 Definition and properties

This is the "classical" way to write a complex number.

Definition 2. We have for all complex number $z \in \mathbb{C}$:

$$z = a + ib \ , \ (a, b) \in \mathbb{R}^2$$

a is called the real part of z and we denote : $a = \mathrm{Re}(z)$
b is called the imaginary part of z and we denote : $b = \mathrm{Im}(z)$

Example 1. Find the real part and the imaginary part of $z = 2 - 3i$.

Definition 3. Affix and image
Each complex number z is associated to a point M in the Cartesian plane $\left(O, \overrightarrow{i}, \overrightarrow{j}\right)$, such that its coordinates are the real part and the imaginary part of z. We say that M is the **image** of z and that z is the **affix** of M.

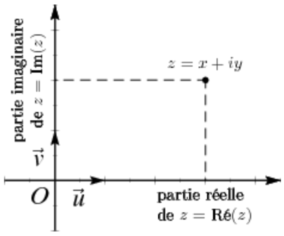


FIGURE 1 – Graphic interpretation of a complex number

Example 2. Draw the complex number with affix $2 - 3i$

Remark 1. There is no i in the imaginary part.

Property 1. Two complex numbers are equal z and z' are equal if and only if $\text{Re}(z) = \text{Re}(z')$ and $\text{Im}(z) = \text{Im}(z')$.

Example 3. Solve the equation : $(x + 2i)(1 + 3i) = 2i(1 + xi)$ where x is a real.

Addition and multiplicative properties are the same as in $\mathbb{R}$ knowing that $i^2 = -1$.

Example 4. Find the rectangular form of $(1 + 2i)(2 - 3i)$.

1.2 Complex conjugate

Definition 4. Let $z = a + ib$ be a complex number, then its conjugate is : $\bar{z} = a - ib$.

Remark 2. Conjugate
Graphically the point M' of affix $\bar{z}$ and the point M of affix z are symmetrical over the x-axis.

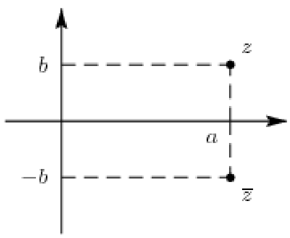


FIGURE 2 – Complex conjugate

Remark 3. In Physics, if $i(t) = I_0 \cos(\omega t)$, the complex intensity is denoted by $\underline{I} = I_0 e^{j\omega t}$ and the conjugate $\underline{I}$ is denoted by $\underline{I}^*$.

Example 5. Find the complex conjugate of $1 + i(2 + 3i)$.

We could simplify expressions with z and $\bar{z}$ knowing that :

$$z + \bar{z} = 2\,\text{Re}(z) \text{ et } z - \bar{z} = 2i\,\text{Im}(z)$$

Property 2.
$\overline{z+z'} = \overline{z} + \overline{z'}$
$\overline{zz'} = \overline{z}\overline{z'}$
$\overline{\left(\frac{z}{z'}\right)} = \frac{\overline{z}}{\overline{z'}}$

Example 6. Prove that $\overline{zz'} = \overline{z}\overline{z'}$.

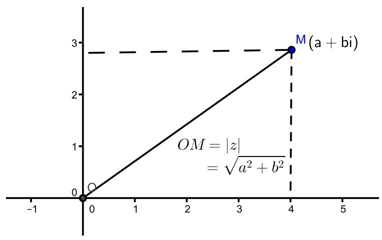
Remark 4. The rectangular form of $\frac{z}{z'}$ is got by multipliying the numerator and the denominator by the conjugate of z', $\overline{z'}$.

Example 7. Find the rectangular form of $\frac{1+2i}{3i+2}$.

1.3 Modulus

Definition 5. The modulus of $z = a + bi$, with a and b two real numbers is equal to $\sqrt{a^2+b^2}$, we denote it by $|a+bi|$.

Remark 5. Let z be the affix of M. $|z|$ is the distance OM.



Example 8. Find the modulus of $2-5i$.

Property 3. Relation between modulus et conjugate : $|z|^2 = z\overline{z}$

Example 9. Solve the equation : $z(\overline{z}+1) = z+2+i$. Find $M(z)$ such that $(\overline{(}z)+2-3i)(z+2+3i) = 4$.

Theorem 1. For all $z, z' \in \mathbb{C}$, $|zz'| = |z|.|z'|$, $\quad \left|\frac{z}{z'}\right| = \frac{|z|}{|z'|} \quad$ et $\quad |z+z'| \leqslant |z| + |z'|$

Example 10.

1. Compute the modulus of $\frac{1-i}{i+\sqrt{3}}$
2. Prove that $\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}$

2 Rectangular and polar forms

2.1 Trigonometric formulae

2.1.1 Angles

Property 4.

$\cos(-x) = \cos(x)$	$\sin(-x) = -\sin(x)$
$\cos(\pi - x) = -\cos(x)$	$\sin(\pi - x) = \sin(x)$
$\cos(\pi + x) = -\cos(x)$	$\sin(\pi + x) = -\sin(x)$
$\cos(\frac{\pi}{2} - x) = \sin(x)$	$\sin(\frac{\pi}{2} - x) = \cos(x)$
$\cos(\frac{\pi}{2} + x) = -\sin(x)$	$\sin(\frac{\pi}{2} + x) = \cos(x)$

2.1.2 Fundamental values

Property 5.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	

2.2 Polar form

Definition 6. For all complex number $z = a + bi \in \mathbb{C}$, there exists two real numbers ρ and θ such that :

$$z = \rho\left(\cos(\theta) + i\sin(\theta)\right)$$

thus :

$$z = [\rho; \theta]$$

To find the polar form knowing the rectangular one

- ρ is called the modulus of z and we have : $\rho = |z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$
- θ is called the argument of z andθ is denoted by $arg(z)$ and defined by :
$\begin{cases} \cos(\theta) = \dfrac{a}{\rho} \\ \sin(\theta) = \dfrac{b}{\rho} \end{cases}$ The argument of z can increase by any integer multiple of 2π and still give the same angle.

Example 11. Find the polar form of $1 - i\sqrt{3}$.

Corollary 1. Two complex numbers, written in their polar form, are equal if and only if they have the same modulus and the same argument at 2π.

Example 12. Write this property using mathematical symbols.

In electricity we often use the tangente function and write $\theta = \tan^{-1} \dfrac{\operatorname{Im} z}{\operatorname{Re} z}$ at π .

Example 13. Find an approximation for the argument of $-1 + 2i$.

Theorem 2.
Whatever are θ and θ' in $\mathbb{R}^2$ we have :

$$(\cos(\theta) + i\sin(\theta))(\cos(\theta') + i\sin(\theta')) = \cos(\theta + \theta') + i\sin(\theta + \theta')$$

Example 14. Prove this theorem.

Corollary 2.
For all $z, z' \in \mathbb{C}$, $\arg(zz') = \arg(z) + \arg(z')$ and $\arg\left(\dfrac{z}{z'}\right) = \arg(z) - \arg(z')$

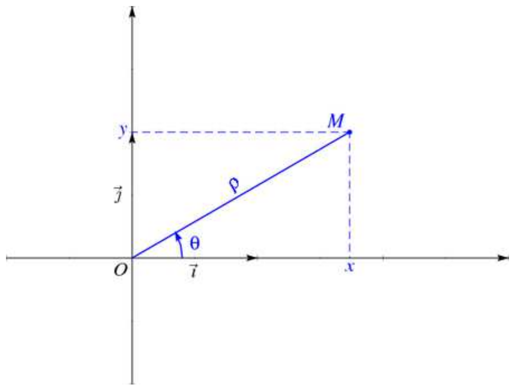
Corollary 3.
For all $\theta \in \mathbb{R}$ and for all $n \in \mathbb{N}$, we have :

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

This formula is known as De Moivre's formulae.

2.3 Geometrical interpretation

We saw before that ρ is the distance OM.
θ is the angle$(\vec{i}, \overrightarrow{OM})$



Remark 6. The geometrical interpretation for θ is useful to find an argument of z.

Example 15. Let's find an argument of 1,i, -1 and $-i$.

2.4 Complex exponential

Definition 7. The great idea of Euler was to define the complex exponential by : for all $\theta \in \mathbb{R}$:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Example 16. Find the polar form of 1, i, -1 and $-i$.

This definition is due to the fundamental property checked by the complex exponential (the same property as for the real exponential) :

Theorem 3. For all $\theta, \theta' \in \mathbb{R}$ we have : $e^{i\theta}.e^{i\theta'} = e^{i(\theta+\theta')}$

Corollary 4. For all $\theta, \theta' \in \mathbb{R}$ we have : $\frac{1}{e^{i\theta}} = e^{-i\theta}$; $\frac{e^{i\theta}}{e^{i\theta'}} = e^{i(\theta-\theta')}$

Corollary 5. For all $\theta \in \mathbb{R}$ and for all $n \in \mathbb{N}$, we have : $\left(e^{i\theta}\right)^n = e^{in\theta}$. With the polar form we get :

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

This formula is called De Moivre's formula.

Example 17. Justify that this corollary is a consequence of the previous properties.

Another fundamental result is :

Theorem 4. For all $\theta \in \mathbb{R}$, $\overline{e^{i\theta}} = e^{-i\theta}$

This theorem is useful to compute the modulus of complex numbers which are a sum of two complex exponential thanks to the formula : $|z|^2 = z\overline{z}$

2.5 Polar form of complex numbers

Definition 8. Let z be a complex number : we set $r = |z|$ and $\theta = \arg z$. Thus we get :

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

This way of writting $z = re^{i\theta}$ is called the polar form of the complex number z.

Example 18. Find the polar form of : $z = \frac{1}{2} + \frac{1}{2}i$

Property 6. Let's define $z_1 = \rho_1 e^{i\theta_1}$ and $z_2 = \rho_2 e^{i\theta_2}$. We have :

$$z_1 z_2 = [\rho_1\rho_2; \theta_1 + \theta_2]$$
$$\frac{z_1}{z_2} = \left[\frac{\rho_1}{\rho_2}; \theta_1 - \theta_2\right]$$
$$z_1^n = [\rho_1^n; n\theta_1]$$

Corollary 6. For all $z, z' \in \mathbb{C}$, $\arg(zz') = \arg(z) + \arg(z')$ and $\arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z')$

Property 7. Euler's formulae.

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

2.6 Linearization

To linearize is to transform a product as a sum. In particular we express $\cos^n(x)$ or $sin^n(x)$ a sa sum of $\cos(nx)$ and of $\sin(nx)$.
This is very useful in integral calculus. To linearize the cosine and the sine function, we distinguish several steps :

- We use Euler's formulae to write the sine and the cosine.
- We develop the expression using Newton's formula : $(a+b)^n = \sum_{k=0}^{k=n} \begin{pmatrix} n \\ k \end{pmatrix} a^k b^{n-k}$
- Then we group terms in pair to make appear sines and cosines with Euler's formulae.

Example 19.
Linearize $\cos^3(x)$.

3 Square roots of a complex number

Look carefully there is an "s", as for each complex number there exists two square roots. As we have two square roots, we can't use the usual notation $\sqrt{}$, known for reals
The notation $\sqrt{}$ deals with the square root function $x \in \mathbb{R}^+ \mapsto \sqrt{x} \in \mathbb{R}^+$, is the inverse function bijection of the function $x \in \mathbb{R}^+ \mapsto x^2 \in \mathbb{R}^+$. The square root function is a bijection, thus there exists a **unique** square root for all positive real number. This explains why we don't use this notation for complex numbers. Thus we write the word "the square roots of z are $\dots$ and $\dots$".

3.1 With the rectangular form

Let $z = a + ib$, $(a, b) \in \mathbb{R}^2$ be any complex number.
Looking for the square roots of z means to solve this equation $Z^2 = z$, with Z a complex number, $Z = x + iy$ with $(x, y) \in \mathbb{R}^2$. Thus we get :

$$Z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

Due to the equality :$Z^2 = z$, we get

$$\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases}$$

as two complex numbers are equal if and only if their real and imaginary parts are equal. Moreover we coul add the equality of modulus so

$$\begin{aligned} &\left|Z^2\right| = |Z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = |z| = \sqrt{a^2+b^2} \\ &\Leftrightarrow x^2 + y^2 = \sqrt{a^2+b^2} \end{aligned}$$

This equality is nice provided that a^2+b^2 is a perfect square thus $\sqrt{a^2+b^2}$ will be an integer. Finally we get a system of three equations :

$$\begin{cases} x^2 - y^2 = a \\ 2xy = b \\ x^2 + y^2 = \sqrt{a^2+b^2} \end{cases}$$

Adding the first and the third one, we get :

$$2x^2 = a + \sqrt{a^2+b^2} \Leftrightarrow x = \pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$$

Don't be afraid as in our examples $\sqrt{a^2+b^2}$ is an integer, consequently we get an easier expression for x.
Now x^2 is known, we'll get two expressions for y thanks to the second equation or thanks to the difference between the third and the first equation. Then using the signs of s and y, we get :

$$y = \frac{b}{2x} = \pm\frac{b}{2\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}}$$

or

$$2y^2 = \sqrt{a^2+b^2} - a \Leftrightarrow y = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$$

so we have our two solutions for the equation $Z^2 = z$. Those are the two square roots of z. So the square roots of z are :

$$\begin{cases} x = +\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} \\ y = +\frac{b}{2\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}} \end{cases} \text{et} \begin{cases} x = -\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} \\ y = -\frac{b}{2\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}} \end{cases}$$

Example 20. Find the square roots of $z = 3 - 4i$.

3.2 With the polar form

This method is only possible if the argument of the unknown complex number z is a well-known angle, for instance $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, or a multiple. Let's focus on this method :

- Let's consider a complex number $z_1 = \rho_1 e^{i\theta_1}$, and we look for $z = \rho e^{i\theta}$ such that $z^2 = z_1$ (E).
- We identify as two complex numbers are equal if and only if they have the same modulus and the same argument(E). (Corollary 2)

- Thus we get ρ and θ in function of ρ_1 and θ_1.
- Our goal is to find those two values.

Example 21. Find square roots of
$$z_1 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$
.

4 n-th roots of a complex number

4.1 n-th roots of the unity

Looking for the n-th order roots of 1 is to solve $z^n = [\rho^n, n\theta] = 1$. Using the trigonometrical method, we get solutions of the form
$\rho = 1$ and $\theta = 2k\pi/n$ where $k \in 0, ..., n-1$.

Example 22. Let's find the third roots of unity.

4.2 n-th roots for any complex numbers

We'll be able to compute the n-th roots of a complex number (exact values) only if its argument is a well-known angle. (if not this is a tricky problem) Let's assume that this complex number is $z = [\rho, \theta]$, with θ a well-known angle. Then, to find the n-th roots of z is to solve $Z^n = z$ with $Z = [r, \phi]$. Thus solutions are : $Z_k = [\sqrt[n]{\rho}, \frac{\theta}{n} + \frac{2k\pi}{n}]$ where $k \in 0, ..., n-1$.

Example 23. Find the fifth roots of $z = 1 + i$.

5 To solve a complex equation

5.1 Quadratic equation

In this section we focus on the compew equation :

$$az^2 + bz + c = 0, (a, b, c, z) \in \mathbb{C}^* \times \mathbb{C}^3$$

The problem is the notation to write square roots of this discriminant. Due to rhe previous section we know there exists two square roots for ths discriminant.

Theorem 5. The discriminant for the complex equation $az^2 + bz + c = 0$, is defined by :

$$\Delta = b^2 - 4ac$$

There is no ordering relation in $\mathbb{C}$, this means that we cannot compare two complex numbers z_1 and z_2 to know if $z_1 < z_2$ or si $z_1 > z_2$. By the way we can't speak about the sign of Δ thus we have only one case :

$$z_1 = \frac{-b - \sqrt{\Delta}}{2a}$$
$$z_2 = \frac{-b + \sqrt{\Delta}}{2a}$$

Of course we can't write $\sqrt{\Delta}$ as Δ is a complex number. So we use the notation to keep in mind that the formulae are similar to formulae used for reals. Moreover, we choose one of the two square roots of Δ, z_1 or z_2.

Example 24. Solve the complex equation :

$$z^2 + (1 - 4i)z + i - 5 = 0$$

5.2 Exercises

Exercise 1.
Find the rectangular form of those complex numbers :

1. $(2 + 3i)(5 - i)$; $(-1 + 2i)(3i + 5)$
2. $(1 + 2i)^2$; $(2 - i)^2$
3. $\frac{1 + 2i}{1 + 3i}$; $\frac{2i}{2 - 3i}$
4. z such that $\frac{1}{z} = \frac{1}{R} + iC\omega$

Exercise 2.
Write in function of $\bar{z}$: $\overline{\left(\frac{2iz + 3}{(5z + 2i)(z + 1)}\right)}$

Exercise 3.
Find two complex numbers z and z' such that $|z + z'| < |z| + |z'|$, and two other complex numbers such that $|z + z'| = |z| + |z'|$

Exercise 4.
Solve in $\mathbb{C}$ the equation $\frac{2 + i}{2 + z - i} = \frac{2 + 3i}{5 - 2i}$

Exercise 5.
Solve in $\mathbb{C}$ the equation $\frac{3 + 2iz}{2 + 3i} = \frac{-1 + 2i}{i + 3}$

Exercise 6.
Let R, C, L and ω be three real positive numbers. In electricty, we define

1. $\frac{1}{z} = \frac{1}{jL\omega} + \frac{1}{R}$

2. $\frac{1}{z} = \frac{1}{jL\omega} + jC\omega$

Compute the rectangular form of the previous numbers.

Exercise 7.
Write the rectangular form of those complex numbers : $e^{i\frac{\pi}{3}}, e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{-i\frac{\pi}{3}}, e^{-i\frac{\pi}{4}}, e^{-i\frac{\pi}{2}}, e^{-i\pi}$

Exercise 8.
Write the exponential form of those complex numbers : $i, -i, 1+i, 1-i, \frac{1}{i}, \frac{1+i}{1-i}, \sqrt{3}+i, \sqrt{3}-i, -e^{i\theta}$,
$\cos\theta - i\sin\theta, \sin\theta - i\cos\theta$

Exercise 9.
Let R, C, L and ω be real positive numbers. In electricty, we define

1. $\frac{1}{z} = \frac{1}{jL\omega} + \frac{1}{R}$
2. $\frac{1}{z} = \frac{1}{jL\omega} + jC\omega$

Compute the polar form of the previous numbers.

Exercise 10.
Calculate $z = \left(1+\sqrt{3}i\right)^{13}$ et $(\sqrt{3}-i)^{(\text{our year})}$

Exercise 11.
Let's put $z_1 = 1+i$, $z_2 = 1+i\sqrt{3}$ and $z_3 = z_1 z_2$.

1. Find the argument and the modulus of z_1, z_2, z_3.
2. Let's deduce the exact values of $\cos\frac{7\pi}{12}$ and $\sin\frac{7\pi}{12}$.

Exercise 12.
Let's define $z_1 = 2\sqrt{6}\,(1+i)$ and $z_2 = \sqrt{2}\left(1+i\sqrt{3}\right)$

1. Compute the complex number $\frac{z_1}{z_2}$, use its rectangular form.
2. Calculate the argument and the modulus of $z_1, z_2, \frac{z_1}{z_2}$
3. Let's deduce $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$.

Exercise 13.
Let's define $z = e^{i\phi} + e^{i\psi}$.
Prove that $z = e^{i\frac{\phi-\psi}{2}}[e^{-i\frac{\phi-\psi}{2}} + e^{i\frac{\psi-\phi}{2}}]$ and calculate $|z|$.

Exercise 14.
Linearize and find antiderivatives for those functions :

1. $\cos^5 x$
2. $\cos^2 x \sin^3 x$

Exercise 15.
Find square roots of $z_1 = -15 + 8i$, $1 + i$, $-i$, $\sqrt{3} - i$, $z_2 = 4i - 3$, $z_3 = -16 - 30i$.

Exercise 16.
Find the third roots of $2 - 2i$.

Exercise 17.
For all $z \in \mathbb{C}$ we put $P(z) = z^4 - 1$

1. Factorize $P(z)$
2. Let's deduce solutions of this equation $P(z) = 0$
3. Let's deduce solutions of this equation $\left(\dfrac{2z+1}{z-1}\right)^4 = 1$

Exercise 18.
Find the fourth order roots of 81 and -81.

Exercise 19.
Solve in $\mathbb{C}$ those equations with z unknown.

1. $z^2 + (5 - 11i)\, z - 22 - 29i = 0.$
2. $z^4 - 15(1 + 2i)z^2 - 88 + 234i = 0.$

Exercise 20.
Solve in $\mathbb{C}$, the equation : $z^4 = \dfrac{1 - i}{1 + i\sqrt{3}}$.

Exercise 21. (optional)
Let $n \in \mathbb{N}^*$,

1. Solve $z^{2n} + z^n + 1 = 0.$
2. Solve $(z - 1)^n = (z + 1)^n$, and prove that the solutions are purely imaginary numbers.

Exercise 22. (optional)
Solve in $\mathbb{C}$ the following equation with z unknown.

$$z^4 + 2\lambda^2 z^2 (1 + \cos\theta) \cos\theta + \lambda^4 (1 + \cos\theta)^2 = 0$$

where $\lambda \in \mathbb{R}$, and $\theta \in [0; \pi]$.