FUNCTIONS OF TWO VARIABLES

In this section, $\mathcal{A}$ is a part of $\mathbb{R}^2$, it could be a disk, a parabola, an ellipse...

1 Generalities

1.1 Functions from $\mathbb{R}^n$ to $\mathbb{R}^p$

We already studied in mathematics real-valued functions of one variable (one variable and its image is a real number), and we may also haver n variables and its image can be a vector of length p thus we have a function from $\mathbb{R}^n$ to $\mathbb{R}^p$.

Example 1. Find n and p in the following examples :

1. we consider the trajectory of a point M in space with those equations $\begin{cases} x(t) = \cos t \\ y(t) = 3\sin t \\ z(t) = 2t - 3 \end{cases}$
2. the vector field $\vec{V}(x + 2y; 3y + z; x^2 - xy)$
3. the scalar potential $f(x, y, z) = -2x + 3y + 2z$

Definition 1. A function from $\mathbb{R}^n$ to $\mathbb{R}^p$ is defined as follows :

$$(x_1, x_2, ..., x_n) \to (f_1(x_1, x_2, .., x_n), f_2(x_1, x_2, .., x_n), ..., f_p(x_1, x_2, ..., x_n))$$

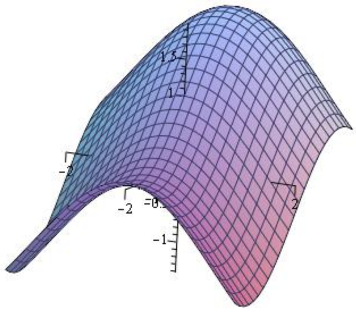
We will focus on the case where $n = 2$ et $p = 1$. It is easy to generalize when $n > 2$ by adding coordinates

1.2 Functions of two real variables

Let's denote $\mathcal{F}(A, \mathbb{R})$, the sset of functions from A to $\mathbb{R}$. A function of two variables is defined as follows :

$$f : \begin{array}{ccc} A & \to & \mathbb{R} \\ (x, y) & \mapsto & z = f(x, y) \end{array}$$

For instance, the relationship $PV = nRT$ for ideal gases, allows to express T in function of P and V. We thus have $T = f(P, V)$ with $f(x, y) = \dfrac{xy}{nR}$.
By the graph of a function of two real variables, we mean the graph of the equation $z = f(x, y)$ which is a surface in $\mathbb{R}^3$.



To draw a surface, softwares use level set : softwares trace the lines C_k where the variables z of $M(x, y, f(x, y))$ is constant and equal to k.

Example 2. For the function $f(x, y) = x^2 + y^2$, trace level set C_k in the plane for $k \in \{0; 1; 2; 3\}$. Taking successively $x = 0$ and $y = 0$, and $f(x, y) = k$, deduce the sketch of the surface

Definition 2. Partial functions of $\mathbb{R}^2$
Let's consider $f : \mathcal{A} \to \mathbb{R}$ and $M_0(x_0, y_0) \in \mathcal{A}$. Partial functions of f at the point M_0 are real valued functions of one variable. Indeed partial functions deal with the simplest case where only one of the two independent variable is changing and the other is held constant :

$$f_{y_0} : x \mapsto f(x, y_0)$$
$$f_{x_0} : y \mapsto f(x_0, y)$$

Example 3. Let's consider $f(x, y) = \dfrac{(2 - y)\cos(xy)}{1 + x^2}$.
Determine its partial functions f at the point $(0; 1)$.

2 Limits, continuity at a point

2.1 Definitions and properties

Let f be a real-valued function defined on a set U of $\mathbb{R}^2$ and M_0 a point of $\mathbb{R}^2$. We denote $d(A, B)$ the distance between the points A and B,

$$AB = d(A, B) = ||\vec{AB}|| = \sqrt{x^2 + y^2}$$

with $x = x_b - x_a$ and $y = y_b - y_a$

Definition 3. Finite Limit
We say that $\lim\limits_{M \to M_0} f(M) = l$ if :
for all non zero distance ε (as small as possible), there always exists an open disk of centre M_0, such that for all point M belonging both to the disk and the domain of definition, the gap between $f(M)$ and l is lower than ε. Thus :

$$\forall \varepsilon > 0, \exists \alpha > 0 / \forall M \in U, d(M_0, M) < \alpha \Rightarrow |f(M) - l| \leqslant \varepsilon$$

Example 4. Evaluate $\lim_{M \to O} f(M)$ with $f(M) = f(x, y) = \sin\left(x^2 + y^2\right)$.

Definition 4. Infinite limit
We say that $\lim_{M \to M_0} f(M) = +\infty$ if and only if : for all real number A (as big as we want), there always exists an open disk of centre M_0, such that for all point M belonging both to the disk and the domain of definition, $f(M)$ is greater than A.

$$\forall A \in \mathbb{R}, \exists \alpha > 0 / \forall M \in U, d(M_0, M) < \alpha \Rightarrow f(M) \geqslant A$$

Definition 5. Infinite limit
We say that $\lim_{M \to M_0} f(M) = -\infty$ if and only if : for all real number A ($A < 0$ and $|A|$ as big as we want), there always exists an open disk of centre M_0, such that for all point M belonging both to the disk and the domain of definition, $f(M)$ is lower than A.

$$\forall A \in \mathbb{R}, \exists \alpha > 0 / \forall M \in U, d(M_0, M) < \alpha \Rightarrow f(M) \leqslant A$$

Example 5. Evaluate $\lim_{M \to 0} f(M)$ with $f(M) = f(x, y) = \dfrac{1}{x^2 + y^2}$.

Theorems studied for functions of one variable can be transposed for functions of two variables. In particular we have the sandwich theorem :

Theorem 1. Let's consider $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that for all (x, y) in the neighborhood of $(x_0, y_0), we have |f(x, y)| \leqslant g(x, y)$ then : if $\lim_{(x,y) \to (x_0, y_0)} g(x, y) = 0$ then $\lim_{(x,y) \to (x_0, y_0)} f(x, y) = 0$

Definition 6. f is said to be continuous at (x_0, y_0) if and only if : $\lim_{M \to M_0} f(M) = f(M_0)$.

2.2 How to compute a limit

Let M_0 be a point of the frontier of the domain of definition or outside the domain

2.2.1 The function has alimit at M_0

To compute a limit we may use inequalities.

Example 6.
Let's define f by $f(x; y) = \dfrac{xy^2}{x^2 + y^2}$.

1. Prove that $|xy| \leqslant \dfrac{1}{2}(x^2 + y^2)$.
2. Deduce that $|f(x, y)| \leqslant \dfrac{|y|}{2}$.
3. and conclude for the limit of f at $(0, 0)$.

Remark 1. We may also use polar coordinates setting : $x = r\cos\theta$ et $y = r\sin\theta$:

$$((x, y) \longrightarrow (0, 0) \Longleftrightarrow (r \to 0 \;\text{ and }\; \theta \text{ any})$$

Example 7. Solve the previous example using polar coordinates.

2.2.2 The function has no limit at M_0

Proposition 1 (Path rules).
Let u and v be two continuous function such that $\lim\limits_{x \to x_0} u(x) = y_0$ and $\lim\limits_{y \to y_0} v(y) = x_0$. If f has a limit l at (x_0, y_0) then we get $\lim\limits_{x \to x_0} f(x, u(x)) = l$ et $\lim\limits_{y \to y_0} f(v(y), y) = l$

Remark 2. Be careful as the converse is false : if f admits equal limits on several paths, it does not mean that f has a limit.

By contraposition,

Proposition 2.
If the limits on two different paths are not equal then we may deduce that f has no limit at (x_0, y_0).

We will often use this property to prove that f has no limit.

Example 8. Prove that the function defined by : $f(x, y) = \dfrac{xy}{x^2 + y^2}$ has no limit at $(0, 0)$.

Remark 3. We may also use this change of variables $x = r\cos\theta$ and $y = r\sin\theta$ and prove that $f(x, y)$ depends on θ as r approaches 0.

Example 9. Solve the previous example with this method.

3 Partial derivatives

3.1 Definition

Let U be an open set of $\mathbb{R}^2$ and f be a function from U to $\mathbb{R}$. f admits partial derivatives at the point $M_0(x_0, y_0)$ if both $\lim\limits_{t \to 0} \dfrac{f(x_0 + t, y_0) - f(x_0, y_0)}{t}$ and $\lim\limits_{t \to 0} \dfrac{f(x_0, y_0 + t) - f(x_0, y_0)}{t}$ are finite. This is denoted by :

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t}$$
$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0, y_0 + t) - f(x_0, y_0)}{t}$$

Values taken by partial derivatives are those taken by the derivatives of the partial functions f_{x_0} and f_{y_0}.

Example 10. Compute the partial derivatives of $f(x,y) = x^2y^8 + e^x$.

Example 11.
Let f be the function of $\mathbb{R}^2$ defined by :

$$\begin{cases} f(x,y) = & (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} \quad \text{si } (x,y) \neq (0,0) \\ f(0,0) = & 0 \end{cases}$$

Let's determine $\dfrac{\partial f}{\partial x}(0,0)$

3.2 Functions of differentiability class $\mathcal{C}^1$

Let f be a function from U to $\mathbb{R}$ such that for all points in U its partial derivatives exist. f is said of differentiability class $\mathcal{C}^1$ on U if and only the partial derivatives of f are continuous on U. We denote by $\mathcal{C}^1(U)$ the set of all functions of differentiabilty class $\mathcal{C}^1$ on U. This is a sub-vector space of $\mathcal{F}(U,\mathbb{R})$.

Example 12.
Id the previous function 11 of differentiability class C^1 on $\mathbb{R}$?

3.3 Partial derivatives for composition

Property 1.
Let f be a function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(X,Y)$ and ϕ a function $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $\phi(x,y) = (\phi_1(x,y), \phi_2(x,y))$.
Let's define g by $g(x,y) = f(\phi(x,y))$.
We thus get :

$$\frac{\partial g}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial X}(\phi(x_0,y_0))\frac{\partial \phi_1}{\partial x}(x_0,y_0) + \frac{\partial f}{\partial Y}(\phi(x_0,y_0))\frac{\partial \phi_2}{\partial x}(x_0,y_0)$$

$$\frac{\partial g}{\partial y}(x_0,y_0) = \frac{\partial f}{\partial X}(\phi(x_0,y_0))\frac{\partial \phi_1}{\partial y}(x_0,y_0) + \frac{\partial f}{\partial Y}(\phi(x_0,y_0))\frac{\partial \phi_2}{\partial y}(x_0,y_0)$$

By abuse of notations we get :

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial X}\frac{\partial \phi_1}{\partial x} + \frac{\partial f}{\partial Y}\frac{\partial \phi_2}{\partial x} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial x}$$

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial X}\frac{\partial \phi_1}{\partial y} + \frac{\partial f}{\partial Y}\frac{\partial \phi_2}{\partial y} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial y}$$

Example 13.
Calculate $\dfrac{\partial f}{\partial y}$ avec $f(x,y) = g(x^2+2y, 3xy)$.

We will use change of variables for partial differential equation.

3.4 Second order Partial derivatives

Let's consider f a function of differentiability class $\mathcal{C}^1$ defined ona n open set U of $\mathbb{R}^2$. If the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are of differentiability class $\mathcal{C}^1$ on U, f is said of differentiability class $\mathcal{C}^2$ on U. Secnd order partial derivatives are denoted by $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ et $\frac{\partial^2 f}{\partial y \partial x}$.

Theorem 2 (Schwarz). If f is of differentiabily class $\mathcal{C}^2$ on U then we have : $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

4 Differential

4.1 Differentiable Function

Definition 7. Differentiable Function
If there exists two real numbers m and p, and a disk $\mathcal{D}$ of center (x_0, y_0) such that for all $(h, k) \in \mathcal{D}$:

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + mh + pk + \parallel (h, k) \parallel \phi(x_0 + h, y_0 + k)$$

with $\lim_{(h,k)\to(0,0)} \phi(x_0 + h, y_0 + k) = 0$
then we say that :

- the function f is **differentiable** at (x_0, y_0) and $m = \frac{\partial f}{\partial x}(x_0, y_0)$ and $p = \frac{\partial f}{\partial y}(x_0, y_0)$.
- $f(x_0, y_0) + h\frac{\partial f}{\partial x}(x_0, y_0) + k\frac{\partial f}{\partial y}(x_0, y_0) + \parallel (h, k) \parallel \phi(x_0 + h, y_0 + k)$ is the **linear approximation of** f at M_0.

Example 14. Let's give $DL_1((\pi, 1))$ de $f(x, y) = x^3 + y^3 \cos x$.

Remark 4.
If the partial derivatives of f exist at (x_0, y_0), then f is not forcing differentiable at (x_0, y_0).

Example 15.
Let's define f by $f(x, y) = \frac{xy^2}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

1. Prove that f admits two partial derivatives at $(0, 0)$, and compute its partial derivatives.
2. Assuming that f is differentiable at 0, what would be ϕ.
3. Calculate $\phi(x, x)$ for all $x > 0$.
4. Conclude

Proposition 3.
If f is of differentiability class $\mathcal{C}^1$, then f is differentiable, which means that it sufficies that the partial derivatives are continuous to say that f is differentiable.
This implies that the function in the previous example is not of differentiability class$\mathcal{C}^1$

Remark 5.
to be differentiable, it is not necessary that f is of differentiability class C^1

Example 16.
Prove that the function 11 is differentiable but not of differentiability class C^1.

Definition 8. The total differential
Let f be a differentiable at (x_0, y_0).
The map $df(M_0) : (h, k) \mapsto h\frac{\partial f}{\partial x}(M_0) + k\frac{\partial f}{\partial y}(M_0)$ is a linear form on $\mathbb{R}^2$ this means a linear map from $\mathbb{R}^2$ to $\mathbb{R}$.
This is the total differential of f at M_0.

4.2 From mathematics ... to physics

From mathematics ...
To simplify we denote $df(M_0) = df$.
Thus we have $df(h, k) = \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k$
dx and dy are the total differential functions of respectively $(h, k) \mapsto h$ and $(h, k) \mapsto k$.

Example 17. Prove that $dx(h, k) = h$ and $dy(h, k) = k$.

So we may write :

$$df(h, k) = \frac{\partial f}{\partial x}dx(h, k) + \frac{\partial f}{\partial y}dy(h, k)$$

thus $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ with dx and dy maps.

... to physics
When $z = f(x, y)$, the increment of f at the point $M(x_0, y_0)$ is the variable Δf given by $\Delta f(h, k) = f(x_0 + h, y_0 + k) - f(x_0, y_0)$. This is equal to the change in z as x changes from x_0 to $x_0 + h$ and y changes from y_0 to $y_0 + h$.
When $z = f(x, y)$, the total differential of f is the variable given by : $df(h, k) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$
The increment theorem states that the increment of f is very close to the total differential of f if Δx and Δy are infinitesimal. Thus, we get an interpretation for the total differential of a function : this is a linear approximation of the increment of f for infinitesimal increments of x and y.

Example 18. Let $S = xy$ be the area of a rectangle of dimensions x et y.
Find the increment for the previous product function, using physics.
A.N. : $x = 10$, $y = 2$, $dx = 0,1$ et $dy = 0,01$.

4.3 Gradient

Definition 9. Let f be a function of differentiability class $\mathcal{C}^1$ on $\mathcal{U}$. At every point M_0, there exists a unique vector, called il the **gradient** of f at M_0, denoted by $\operatorname{grad} f(M_0)$ or $\nabla f(M_0)$ whose coordinates in the standardbasis of $\mathbb{R}^2$ are : $\left(\frac{\partial f}{\partial x}(M_0), \frac{\partial f}{\partial y}(M_0)\right)$.

Example 19. Let's define f by $f(x,y) = x^2 + 3xy$. Let's give its gradient at the point whose coordinates are (2 ;-3).

Remark 6. Let's notice that : $df(h,k) = \operatorname{grad} f \cdot (h,k)$

Proposition 4. Geometrical interpretation Let f be a function such that $f(x_0, y_0) = z_0$ and f is of differentiability class C^1 $\mathbb{R}^2$.

- the gradient of f at $M_0(x_0, y_0)$ is normal in M_0 to the level curve of equation $f(x,y) = z_0$.
- the gradient vector points in the direction of greatest rate of increase of f(x,y)
- the positive rate of the function f is maximal in the direction of the gradient.

Example 20. Let's define f by $f(x,y) = x^2 + y^2$.
Illustrate the previous property at the point $M_0(1;1;f(1;1))$.

Example 21.
We get this property using the following idea :

1. Give the definition of $\vec{u} \cdot \vec{v}$.
2. Let's asume that $||(h,k)||$ is constant and close to 0.
   (a) What can you say about the set of points M such that $\overrightarrow{M_0M} = (h,k)$?
   (b) Find the vectors (h,k) such that $df(h,k)$ is maximal, then minimal and equal to zero.
   (c) Give a justification for the previous property.

Remark 7. This example gives a geometrical justification but as it is based on an approximation it is not really rigorous. It could become using the parametric representation of a level curve. We will speak about it in another chapter.

4.4 Tangent plane of a surface

By analogy with the tangent line defined for a real valued function of one variable, the tangent plane of a surface with equation $f(x,y) = z$ at the point is the plane with the equation :

$$z = f(x_0, y_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0)$$

Prove this formula/

Exercises

Exercise 1.

Give domain of definitions for those functions. Is it possible to extend the following functions by continuity at the point (0,0) ?

1. $f(x, y) = \dfrac{x^2 + y^2}{x}$
2. $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$
3. $f(x, y) = \dfrac{x^2 + y^2}{|x| + |y|}$
4. $f(x, y) = \dfrac{xy}{x^3 + 3y^2}$
5. $f(x, y) = (x + y^2) \sin\left(\dfrac{1}{xy}\right)$
6. $f(x, y) = \dfrac{1 - \cos(xy)}{y^2}$

Exercise 2.

Let's define f by $f(x, y) = \dfrac{x^2 y}{\sqrt{x^4 + y^2}}$ pour $(x, y) \neq (0, 0)$ et $f(0, 0) = 0$.

1. Let D be a straight line through the origin. Prove that the restriction of f to D is continuous at (0,0).
2. Can we deduce that f est continuous at (0,0) ? Is it continuous ?

Exercise 3.

The law for ideal gases is of the shape $PV = nRT$ where n the number of gs molecules, V the volume, T the temperature, P the pressure et R a constant. Prove that : $\dfrac{\partial V}{\partial T} \dfrac{\partial T}{\partial P} \dfrac{\partial P}{\partial V} = -1$

Exercise 4.

Let's consider $f : f(x, y) = \dfrac{x^2 y^2}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

1. Study the continuity of f on $\mathbb{R}^2$.
2. Compute partial derivatives f'_x and f'_y of f. What are the fonctional values $f'_x(0, 0)$ and $f'_y(0, 0)$?
3. Is f of differentiability class $\mathcal{C}^1$ on $\mathbb{R}^2$?

Exercise 5.

Loads engineers building roads are concerned about the penetration of cold in the ground. To compute the temperature T at time t and at depth x (in m) they use the formula : $T = T_0 e^{-\lambda x} \sin(\omega t - \lambda x)$ where T_0, ω, λ are constants. The period of $\sin(\omega t - \lambda x)$ is 24 hours.

1. Calculate and give an interpretation of $\dfrac{\partial T}{\partial t}$ and $\dfrac{\partial T}{\partial x}$.

2. Prove that T satisfies the heat equation of one dimension $\dfrac{\partial T}{\partial t} = k\dfrac{\partial^2 T}{\partial x^2}$ where k is a constant real number.

Exercise 6.
Find a function f of class $\mathcal{C}^1$ on $\mathbb{R}^2$ such that $\dfrac{\partial f}{\partial x}(x, y) = x + y$. This is called a partial differential equation.

Exercise 7.
A function f is said to be harmonic if $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ on its domain of definition. Prove that those functions are harmonic :

1. $f(x, y) = \ln\sqrt{x^2 + y^2}$

2. $g(x, y) = \mathrm{Arc}\tan\dfrac{y}{x}$

3. $h(x, y) = e^{-x}\cos y + e^{-y}\cos x$

Exercise 8.
Calculate the partial derivatives of h : $h(x, y) = f(2xy, x)$.

Exercise 9.
Sove those partial differential equations :

1. $\dfrac{\partial f}{\partial x} - 3\dfrac{\partial f}{\partial y} = 0$ with $u = 2x + y$ and $v = 3x + y$.

2. $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = x^2 + y^2$ using polar coordinates

Exercise 10.
Prove that the Laplace equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ is equivalent to $g''(r) + \dfrac{1}{r}g'(r) = 0$ with $f(x, y) = g(\sqrt{x^2 + y^2})$.

Exercise 11.
In this exercise, c is a strictly positive fixed real number. All the functions studied are real valued functions. We study the partial differential equation (E), called wave equation :

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

with u unknown function of two real variables x and t, of class C^2 on $\mathbb{R}^2$. Let's define g a function of C^2 such that $g(X;Y) = u(\dfrac{X+Y}{2}, \dfrac{Y-X}{2c})$.

1. We set $u(x, t) = g(X, Y)$. State X and Y en fonction de x et t.

2. Calculate $\frac{\partial u}{\partial x}(x,t)$ et $\frac{\partial u}{\partial t}(x,t)$ in function of g.

3. Deduce $\frac{\partial^2 u}{\partial x^2}(x,t)$ et $\frac{\partial^2 u}{\partial t^2}(x,t)$ in function of g.

4. Solve the equation $\frac{\partial^2 g}{\partial X \partial Y} = 0$.

5. Deduce from the previous questions solutions for the wave equation.

6. Prove that the standing wave $v(t,x) = (\sin ckt)(\sin kx)$ satisfies the wave equation.

Exercise 12.
Using Chain rule, compute dz for : $z = x^3 - y^3; x = \frac{1}{t+1}, y = \frac{t}{t+1}$

Exercise 13.
Calculate the total differential of the following functions :

1. $f(x,y) = x^2 e^{xy} + \frac{1}{y^2}$

2. $f(x,y) = \ln\left(x^2 + y^2\right) + x \operatorname{Arctan} y$

3. $f(x,y) = \frac{xy}{x+y}$

Exercise 14.
Using the total differential, calculate an approximation for the increment of f consecutive to the noticed increments x et y :
$f(x,y) = x^2 - 3x^3y^2 + 4x - 2y^3 + 6$ for (x,y) changes from $(-2,3)$ to $(-2,02,3,01)$.

Exercise 15.
The formula given the electrical resistance for an electric material is $R = \frac{l}{\gamma S}$ where l is the length of the rod, S the surface and γ the conductivity.
Let's assume that S, l et γ are known with an error approximation of 10%.

1. Compute in two different ways $d(\ln R)$

2. Let's deduce the margin of error committed on R ?

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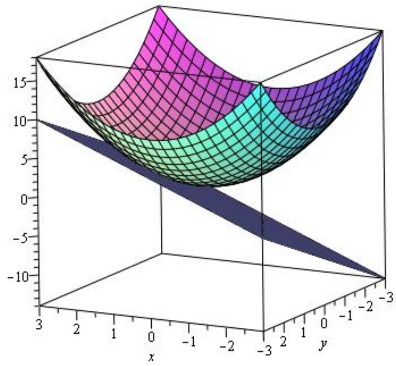
Exercise 16.

Let's define f by $f(x,y) = x^2 + 4y^2$.

1. Compute the gradient of f at $P(2,-1)$.

2. Give an interpretation for this gradient.

Exercise 17.
Soit P be the paraboloid of equation $z = x^2 + y^2$. Give the equation of its tangent plane at the point $M(1, 1, 2)$.



Exercise 18.
Find the points of the hyperboloid of two sheets $x^2 - 2y^2 - 4z^2 = 16$ where the tangent plane is parallel to the plane of equation en $4x - 2y + 4z = 5$.