FUNCTIONS OF TWO VARIABLES

In this section, $\mathcal{A}$ is a part of $\mathbb{R}^2$, it could be a disk, a parabola, an ellipse...

1 Generalities

1.1 Functions from $\mathbb{R}^n$ to $\mathbb{R}^p$

We already studied in mathematics real-valued functions of one variable (one variable and its image is a real number), and we may also haver n variables and its image can be a vector of length p thus we have a function from $\mathbb{R}^n$ to $\mathbb{R}^p$.

Definition 1. A function from $\mathbb{R}^n$ to $\mathbb{R}^p$ is defined as follows :

$$(x_1, x_2, ..., x_n) \to (f_1(x_1, x_2, .., x_n), f_2(x_1, x_2, .., x_n), ..., f_p(x_1, x_2, ..., x_n))$$

Example 1. Find n and p in the following examples :

1. we consider the trajectory of a point M in space with those equations $\begin{cases} x(t) = \cos t \\ y(t) = 3\sin t \\ z(t) = 2t - 3 \end{cases}$
2. the vector field $\vec{V}(x + 2y; 3y + z; x^2 - xy)$
3. the scalar potential $f(x, y, z) = -2x + 3y + 2z$

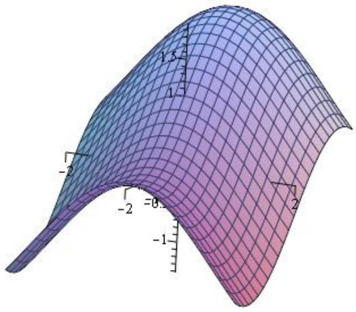
We will focus on the case where $n = 2$ et $p = 1$. It is easy to generalize when $n > 2$ by adding coordinates

1.2 Functions of two real variables

Let's denote $\mathcal{F}(A, \mathbb{R})$, the sset of functions from A to $\mathbb{R}$. A function of two variables is defined as follows :

$$f : \begin{array}{ccc} A & \to & \mathbb{R} \\ (x, y) & \mapsto & z = f(x, y) \end{array}$$

For instance, the relationship $PV = nRT$ for ideal gases, allows to express T in function of P and V. We thus have $T = f(P, V)$ with $f(x, y) = \dfrac{xy}{nR}$.
By the graph of a function of two real variables, we mean the graph of the equation $z = f(x, y)$ which is a surface in $\mathbb{R}^3$.



To draw a surface, softwares use level set : softwares trace the lines C_k where the variables z of $M(x, y, f(x, y))$ is constant and equal to k.

Example 2. For the function $f(x, y) = x^2 + y^2$, trace level set C_k in the plane for $k \in \{0; 1; 2; 3\}$. Taking successively $x = 0$ and $y = 0$, and $f(x, y) = k$, deduce the sketch of the surface

Definition 2. Partial functions of $\mathbb{R}^2$
Let's consider $f : \mathcal{A} \to \mathbb{R}$ and $M_0(x_0, y_0) \in \mathcal{A}$. Partial functions of f at the point M_0 are real valued functions of one variable. Indeed partial functions deal with the simplest case where only one of the two independent variable is changing and the other is held constant :

$$f_{y_0} : x \mapsto f(x, y_0)$$
$$f_{x_0} : y \mapsto f(x_0, y)$$

Example 3. Let's consider $f(x, y) = \dfrac{(2 - y)\cos(xy)}{1 + x^2}$.
Determine its partial functions f at the point $(0; 1)$.

2 Limits, continuity at a point

2.1 Definitions and properties

Let f be a real-valued function defined on a set U of $\mathbb{R}^2$ and M_0 a point of $\mathbb{R}^2$. We denote $d(A, B)$ the distance between the points A and B,

$$AB = d(A, B) = ||\vec{AB}|| = \sqrt{x^2 + y^2}$$

with $x = x_b - x_a$ and $y = y_b - y_a$

Definition 3. Finite Limit
We say that $\lim\limits_{M \to M_0} f(M) = l$ if :
for all non zero distance ε (as small as possible), there always exists an open disk of centre M_0, such that for all point M belonging both to the disk and the domain of definition, the gap between $f(M)$ and l is lower than ε. Thus :

$$\forall \varepsilon > 0, \exists \alpha > 0 / \forall M \in U, d(M_0, M) < \alpha \Rightarrow |f(M) - l| \leqslant \varepsilon$$

Example 4. Evaluate $\lim_{M \to O} f(M)$ with $f(M) = f(x, y) = \sin\left(x^2 + y^2\right)$.

Definition 4. Infinite limit
We say that $\lim_{M \to M_0} f(M) = +\infty$ if and only if : for all real number A (as big as we want), there always exists an open disk of centre M_0, such that for all point M belonging both to the disk and the domain of definition, $f(M)$ is greater than A.

$$\forall A \in \mathbb{R}, \exists \alpha > 0 / \forall M \in U, d(M_0, M) < \alpha \Rightarrow f(M) \geqslant A$$

Definition 5. Infinite limit
We say that $\lim_{M \to M_0} f(M) = -\infty$ if and only if : for all real number A ($A < 0$ and $|A|$ as big as we want), there always exists an open disk of centre M_0, such that for all point M belonging both to the disk and the domain of definition, $f(M)$ is lower than A.

$$\forall A \in \mathbb{R}, \exists \alpha > 0 / \forall M \in U, d(M_0, M) < \alpha \Rightarrow f(M) \leqslant A$$

Example 5. Evaluate $\lim_{M \to 0} f(M)$ with $f(M) = f(x, y) = \dfrac{1}{x^2 + y^2}$.

Theorems studied for functions of one variable can be transposed for functions of two variables. In particular we have the sandwich theorem :

Theorem 1. Let's consider $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that for all (x, y) in the neighborhood of $(x_0, y_0), we have |f(x, y)| \leqslant g(x, y)$ then : if $\lim_{(x,y) \to (x_0,y_0)} g(x, y) = 0$ then $\lim_{(x,y) \to (x_0,y_0)} f(x, y) = 0$

Definition 6. f is said to be continuous at (x_0, y_0) if and only if : $\lim_{M \to M_0} f(M) = f(M_0)$.

2.2 How to compute a limit

Let M_0 be a point of the frontier of the domain of definition or outside the domain.

2.2.1 The function has a limit at M_0

Either the computation ist **"easy"**

Example 6.
Let's define f by $f(x; y) = x^2 + y^2$. Fiind the limit of f at $(0, 0)$

If the computation is not easy, we may use **inequalities**.

Example 7.
Let's define f by $f(x; y) = x\dfrac{1}{x^2 + y^2}$. Prove that f has a limit at $(0, 0)$ using an inequality.

Remark 1. We may also use polar coordinates setting : $x = r\cos\theta$ et $y = r\sin\theta$:

$$((x, y) \longrightarrow (0, 0) \Longleftrightarrow (r \to 0 \text{ and } \theta \text{ any})$$

Example 8. Let's define f by $f(x; y) = \dfrac{xy^2}{x^2 + y^2}$. Prove that f has a limit at $(0, 0)$ using polar coordinates.

2.2.2 The function has no limit at M_0

Proposition 1 (Path rules).
Let u and v be two continuous function such that $\lim_{x \to x_0} u(x) = y_0$ and $\lim_{y \to y_0} v(y) = x_0$. If f has a limit l at (x_0, y_0) then we get $\lim_{x \to x_0} f(x, u(x)) = l$ et $\lim_{y \to y_0} f(v(y), y) = l$

Remark 2. Be careful as the converse is false : if f admits equal limits on several paths, it does not mean that f has a limit.

By contraposition,

Proposition 2.
If the limits on two different paths are not equal then we may deduce that f has no limit at (x_0, y_0).

We will often use this property to prove that f has no limit.

Example 9. Prove that the function defiined by : $f(x, y) = \dfrac{xy}{x^2 + y^2}$ has no limit at $(0, 0)$.

Remark 3. We may also use this change of variables $x = r\cos\theta$ and $y = r\sin\theta$ and prove that $f(x, y)$ depends on θ as r approaches 0.

Example 10. Solve the previous example with this method.

3 Partial derivatives

Partial derivatives calculate the rate of change of a function of several variables with respect to one of those variables while holding the other variables fixed or constant.
In other words, a partial derivative allows only one variable to vary (change) at a time and helps us to analyze surfaces for minimum and maximum points. If x changes we denote it by $\dfrac{\partial f}{\partial x}$ and if y varies we denote $\dfrac{\partial f}{\partial y}$.

Example 11. Compute the partial derivatives of $f(x, y) = x^2 y^8 + e^x$.

3.1 Definition

Let U be an open set of $\mathbb{R}^2$ and f be a function from U to $\mathbb{R}$. f admits partial derivatives at the point $M_0(x_0, y_0)$ if both $\lim\limits_{t\to 0} \frac{f(x_0+t, y_0) - f(x_0, y_0)}{t}$ and $\lim\limits_{t\to 0} \frac{f(x_0, y_0+t) - f(x_0, y_0)}{t}$ are finite. This is denoted by :

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{t\to 0} \frac{f(x_0+t, y_0) - f(x_0, y_0)}{t}$$
$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{t\to 0} \frac{f(x_0, y_0+t) - f(x_0, y_0)}{t}$$

Values taken by partial derivatives are those taken by the derivatives of the partial functions f_{x_0} and f_{y_0}.

Example 12.
Let f be the function of $\mathbb{R}^2$ defined by :

$$\begin{cases} f(x,y) = & (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} \quad \text{si } (x,y) \neq (0,0) \\ f(0,0) = & 0 \end{cases}$$

Let's determine $\frac{\partial f}{\partial x}(0,0)$

3.2 Functions of differentiability class $\mathcal{C}^1$

Let f be a function from U to $\mathbb{R}$ such that for all points in U its partial derivatives exist. f is said of differentiability class $\mathcal{C}^1$ on U if and only the partial derivatives of f are continuous on U. We denote by $\mathcal{C}^1(U)$ the set of all functions of differentiabilty class $\mathcal{C}^1$ on U. This is a sub-vector space of $\mathcal{F}(U, \mathbb{R})$.

Example 13.
Is the previous function of differentiability class C^1 on $\mathbb{R}$?

3.3 Partial derivatives for composition

Property 1.
Let f be a function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(u,v)$ and ϕ a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $\phi(x,y) = (u(x,y), v(x,y))$. Let g be the function defined by $g(x,y) = f(\phi(x,y)) = f(u(x,y), v(x,y)$. Thus we get

$$\frac{\partial g}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial u}(\phi(x_0, y_0))\frac{\partial u}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial v}(\phi(x_0, y_0))\frac{\partial v}{\partial x}(x_0, y_0)$$

$$\frac{\partial g}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial u}(\phi(x_0, y_0))\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial f}{\partial v}(\phi(x_0, y_0))\frac{\partial v}{\partial y}(x_0, y_0)$$

By abuse of notations we get

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

Example 14.
Calculate $\frac{\partial f}{\partial y}$ avec $f(x,y) = g(x^2+2y, 3xy)$.

We will use change of variables for partial differential equation.

3.4 Second order Partial derivatives

Let's consider f a function of differentiability class $\mathcal{C}^1$ defined ona n open set U of $\mathbb{R}^2$. If the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are of differentiability class $\mathcal{C}^1$ on U, f is said of differentiability class $\mathcal{C}^2$ on U. Secnd order partial derivatives are denoted by $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$ et $\frac{\partial^2 f}{\partial y \partial x}$.

Theorem 2 (Schwarz). If f is of differentiability class $\mathcal{C}^2$ on U then we have : $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

4 Differential

4.1 Differentiable Function

Definition 7. Differentiable Function
If there exists two real numbers m and p, and a disk $\mathcal{D}$ of center (x_0, y_0) such that for all $(h,k) \in \mathcal{D}$:

$$f(x_0+h, y_0+k) = f(x_0,y_0) + mh + pk + \| (h,k) \| \, \phi(x_0+h, y_0+k)$$

with $\lim_{(h,k)\to(0,0)} \phi(x_0+h, y_0+k) = 0$
then we say that :

- the function f is **differentiable** at (x_0, y_0) and $m = \frac{\partial f}{\partial x}(x_0,y_0)$ and $p = \frac{\partial f}{\partial y}(x_0,y_0)$.
- $f(x_0,y_0) + h\frac{\partial f}{\partial x}(x_0,y_0) + k\frac{\partial f}{\partial y}(x_0,y_0) + \| (h,k) \| \, \phi(x_0+h, y_0+k)$ is the **linear approximation of** f at M_0.

Example 15. Let's give $DL_1((\pi, 1))$ de $f(x,y) = x^2 + y^2$.

Remark 4.
If the partial derivatives of f exist at (x_0,y_0), then f is not forcing differentiable at (x_0,y_0).

Definition 8. The total differential
Let f be a differentiable at (x_0, y_0).
The map $df(M_0) : (h,k) \mapsto h\dfrac{\partial f}{\partial x}(M_0) + k\dfrac{\partial f}{\partial y}(M_0)$ is a linear form on $\mathbb{R}^2$ this means a linear map from $\mathbb{R}^2$ to $\mathbb{R}$.
This is the total differential of f at M_0.

4.2 From mathematics ... to physics

From mathematics ...
To simplify we denote $df(M_0) = df$.
Thus we have $df(h,k) = \dfrac{\partial f}{\partial x}h + \dfrac{\partial f}{\partial y}k$
dx and dy are the total differential functions of respectively $(h,k) \mapsto h$ and $(h,k) \mapsto k$.

Example 16. Prove that $dx(h,k) = h$ and $dy(h,k) = k$.

So we may write :

$$df(h,k) = \frac{\partial f}{\partial x}dx(h,k) + \frac{\partial f}{\partial y}dy(h,k)$$

thus $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$ with dx and dy maps.

... to physics
When $z = f(x,y)$, the increment of f at the point $M(x_0, y_0)$ is the variable Δf given by $\Delta f(h,k) = f(x_0 + h, y_0 + k) - f(x_0, y_0)$. This is equal to the change in z as x changes from x_0 to $x_0 + h$ and y changes from y_0 to $y_0 + h$.
When $z = f(x,y)$, the total differential of f is the variable given by : $df(h,k) = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$
The increment theorem states that the increment of f is very close to the total differential of f if Δx and Δy are infinitesimal. Thus, we get an interpretation for the total differential of a function : this is a linear approximation of the increment of f for infinitesimal increments of x and y.

Example 17. Let $S = xy$ be the area of a rectangle of dimensions x et y.
Find the increment for the previous product function, using physics.
A.N. : $x = 10$, $y = 2$, $dx = 0,1$ et $dy = 0,01$.

4.3 Gradient

Definition 9. Let f be a function of differentiability class $\mathcal{C}^1$ on U. At every point M_0, there exists a unique vector, called il the **gradient** of f at M_0, denoted by $\mathrm{grad}\, f(M_0)$ or $\nabla f(M_0)$ whose coordinates in the standardbasis of $\mathbb{R}^2$ are : $\left(\dfrac{\partial f}{\partial x}(M_0), \dfrac{\partial f}{\partial y}(M_0)\right)$.

Example 18. Let's define f by $f(x,y) = x^2 + 3xy$. Let's give its gradient at the point whose coordinates are (2 ;-3).

Remark 5. Let's notice that : $df(h,k) = \mathrm{grad}\, f \cdot (h,k)$

Proposition 3. Geometrical interpretation Let f be a function such that $f(x_0, y_0) = z_0$ and f is of differentiability class C^1 $\mathbb{R}^2$.

- the gradient of f at $M_0(x_0, y_0)$ is normal in M_0 to the level curve of equation $f(x,y) = z_0$.
- the gradient vector points in the direction of greatest rate of increase of f(x,y)
- the positive rate of the function f is maximal in the direction of the gradient.

Example 19. Let's define f by $f(x,y) = x^2 + y^2$.
Illustrate the previous property at the point $M_0(1;1;f(1;1))$.

4.4 Scalar Potential

Definition 10.
A vector field is a map from an open sett U of $\mathbb{R}^3$ (resp $\mathbb{R}^2$) to $\mathbb{R}^3$(resp $\mathbb{R}^2$) .

Definition 11.
Let $\overrightarrow{E}$ be a vector field of differentiability $\mathcal{C}^1$ on an open set U de $\mathbb{R}^3$:

$$\overrightarrow{E} : (x,y,z) \to (E_x(x,y,z), E_y(x,y,z); E_z(x,y,z))$$

The curl of the vector field $\overrightarrow{E}$ denoted by $\overrightarrow{\mathrm{rot}}\,\overrightarrow{E}$ defined on U by

$$\overrightarrow{\mathrm{rot}}\,\overrightarrow{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

Definition 12.
Let $\overrightarrow{E}$ be a vector field of differentiability class C^1 on an open set U of $\mathbb{R}^3$. The vector field $\overrightarrow{E}$ comes from a scalar potential if and only if it exists a function f from U to $\mathbb{R}$ called scalar field such that $\overrightarrow{E} = -\overrightarrow{\mathrm{grad}}f$. If such a scalar fiel exists, f is called scalar potential $\overrightarrow{E}$.

Example 20.
$\overrightarrow{g} = -g\overrightarrow{k}$ comes from a scalar potential as $\overrightarrow{g} = -\overrightarrow{\mathrm{grad}}V$ where $V = mgz$.

Theorem 3.
Let U be an open set of $\mathbb{R}^3$ et $\overrightarrow{E}$ a vector field of differentiability class $\mathcal{C}^1$ on U. If $\overrightarrow{\mathrm{rot}}\,\overrightarrow{E} = 0$ and if U is a star open set then $\overrightarrow{E}$ admet un potentiel scalaire.

Example 21.
Prove that the field $\overrightarrow{F}$ comes from a scalar potential and compute it for $\overrightarrow{F}(x,y,z) = (x^2 - yz, y^2 - zx, z^2 - xy)$ in $\mathbb{R}^3$

4.5 Tangent plane to a surface

By analogy with the tangent line defined for function of one real variable, the tangent plane to a surfac, whose equation is $f(x,y) = z$, at the point $M_0(x_0, y_0, z_0)$ is the plane of equation :

$$z = f(x_0, y_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0)$$

Example 22. Prove this formula

Exercises

4.5.1 Domains of definition, level sets, limits, continuity

Exercise 1.
Give domain of definition for those functions. Is it possible to extend the following functions by continuity at the point (0,0) ?

1. $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$
2. $f(x,y) = (x+y^2)\sin\left(\dfrac{1}{xy}\right)$
3. $f(x,y) = \dfrac{xy}{x^2+y^2}$
4. $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$
5. $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

Exercise 2.
Find level sets for those functions

$$f(x,y) = x+y-1 \quad f(x,y) = e^{y-x^2}$$

Exercise 3.
Let's define f by $f(x,y) = \dfrac{x^2 y}{\sqrt{x^4+y^2}}$ pour $(x,y) \neq (0,0)$ et $f(0,0) = 0$.

1. Let D be a straight line through the origin. Prove that the restriction of f to D is continuous at (0,0).
2. Can we deduce that f est continuous at (0,0) ? Is it continuous ?

4.5.2 Partial derivatives : computations and equations

Exercise 4.
The law for ideal gases is of the shape $PV = nRT$ where n the number of gs molecules, V the volume, T the temperature, P the pressure et R a constant. Prove that : $\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P}\dfrac{\partial P}{\partial V} = -1$

Exercise 5.
Let's consider f : $f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

1. Study the continuity of f on $\mathbb{R}^2$.

2. Compute partial derivatives f'_x and f'_y of f. What are the fonctional values $f'_x(0,0)$ and $f'_y(0,0)$?

3. Is f of differentiability class $\mathcal{C}^1$ on $\mathbb{R}^2$?

Exercise 6.
Loads engineers building roads are concerned about the penetration of cold in the ground. To compute the temperature T at time t and at depth x (in m) they use the formula : $T = T_0 e^{-\lambda x} \sin(\omega t - \lambda x)$ where T_0, ω, λ are constants. The period of $\sin(\omega t - \lambda x)$ is 24 hours.

1. Calculate and give an interpretation of $\dfrac{\partial T}{\partial t}$ and $\dfrac{\partial T}{\partial x}$.

2. Prove that T satisfies the heat equation of one dimension $\dfrac{\partial T}{\partial t} = k\dfrac{\partial^2 T}{\partial x^2}$ where k is a constant real number.

Exercise 7.
Find a function f of class $\mathcal{C}^1$ on $\mathbb{R}^2$ such that $\dfrac{\partial f}{\partial x}(x,y) = x + y$. This is called a partial differential equation.

Exercise 8.
A function f is said to be harmonic if $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ on its domain of definition. Prove that those functions are harmonic :

1. $f(x,y) = \ln\sqrt{x^2+y^2}$

2. $g(x,y) = \operatorname{Arc}\tan\dfrac{y}{x}$

3. $h(x,y) = e^{-x}\cos y + e^{-y}\cos x$

Exercise 9.
Calculate the partial derivatives of h : $h(x,y) = f(2xy, x)$.

Exercise 10.
Sove those partial differential equations :

1. $\dfrac{\partial f}{\partial x} - 3\dfrac{\partial f}{\partial y} = 0$ with $u = 2x + y$ and $v = 3x + y$.

2. $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = x^2 + y^2$ using polar coordinates

Exercise 11.
Prove that the Laplace equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ is equivalent to $g''(r) + \dfrac{1}{r}g'(r) = 0$ with $f(x,y) = g(\sqrt{x^2+y^2})$.

Exercise 12.
In this exercise, c is a strictly positive fixed real number. All the functions studied are real valued functions. We study the partial differential equation (E), called wave equation :

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

with u unknown function of two real variables x and t, of class C^2 on $\mathbb{R}^2$. Let's define g a function of C^2 such that $g(X;Y) = u(\frac{X+Y}{2}, \frac{Y-X}{2c})$.

1. We set $u(x,t) = g(X,Y)$. State X and Y en fonction de x et t.
2. Calculate $\frac{\partial u}{\partial x}(x,t)$ et $\frac{\partial u}{\partial t}(x,t)$ in function of g.
3. Deduce $\frac{\partial^2 u}{\partial x^2}(x,t)$ et $\frac{\partial^2 u}{\partial t^2}(x,t)$ in function of g.
4. Solve the equation $\frac{\partial^2 g}{\partial X \partial Y} = 0$.
5. Deduce from the previous questions solutions for the wave equation.
6. Prove that the standing wave $v(t,x) = (\sin ckt)(\sin kx)$ satisfies the wave equation.

4.5.3 Differentials

Exercise 13.
Calculate the total differential of the following functions :

1. $f(x,y) = x^2 e^{xy} + \frac{1}{y^2}$
2. $f(x,y) = \ln\left(x^2 + y^2\right) + x \operatorname{Arctan} y$
3. $f(x,y) = \frac{xy}{x+y}$

Exercise 14.
Using the total differential, calculate an approximation for the increment of f consecutive to the noticed increments x et y :
$f(x,y) = x^2 - 3x^3y^2 + 4x - 2y^3 + 6$ for (x,y) changes from $(-2,3)$ to $(-2.02, 3.01)$.

Exercise 15.
The formula given the electrical resistance for an electric material is $R = \frac{l}{\gamma S}$ where l is the length of the rod, S the surface and γ the conductivity.
Let's assume that S, l et γ are known with an error approximation of 10%.

1. Compute in two different ways $d(\ln R)$
2. Let's deduce the margin of error committed on R ?

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4.5.4 Gradient and vector potential

Exercise 16.

Let's define f by $f(x, y) = x^2 + 4y^2$.

1. Compute the gradient of f at $P(2, -1)$.
2. Give an interpretation for this gradient.

Exercise 17.
Find coordinates for the gradient of f where f is the vector field : $f(x, y, z) = xyz\sin(xy)$

Exercise 18.
Find all the vector potentials for those vector fields

1. $F(x, y, z) = (2xy + z^3, x^2, 3xz^2)$ in $\mathbb{R}^3$
2. $F(x, y, z) = (\dfrac{-y}{(x-y)^2}, \dfrac{x}{(x-y)^2}$ defined on $U = \{(x, y) \in \mathbb{R}^2 x > y\}$

4.5.5 Tangent plane

Exercise 19.
Soit P be the paraboloid of equation $z = x^2 + y^2$. Give the equation of its tangent plane at the point $M(1, 1, 2)$.