# Integration and anti-derivatives

Calculus deals principally with two geometric problems :

- (i) The problem of finding SLOPE of the tangent line to the curve, is studied by the limiting process known as differentiation and
- (ii) Problem of finding the AREA of a region under a curve is studied by another limiting process called Integration.

Actually integral calculus was developed into two different directions over a long period independently.

- (i) Leibnitz and his school of thought approached it as the anti derivative of a differentiable function.
- (ii) Archimedes, Eudoxus and others developed it as a numerical value equal to the area under the curve of a function for some interval.

However as far back as the end of the 17th century it became clear that a general method for solution of finding the area under the given curve could be developed in connection with definite problems of integral calculus.

# 1 Riemann Integral

## 1.1 Introduction

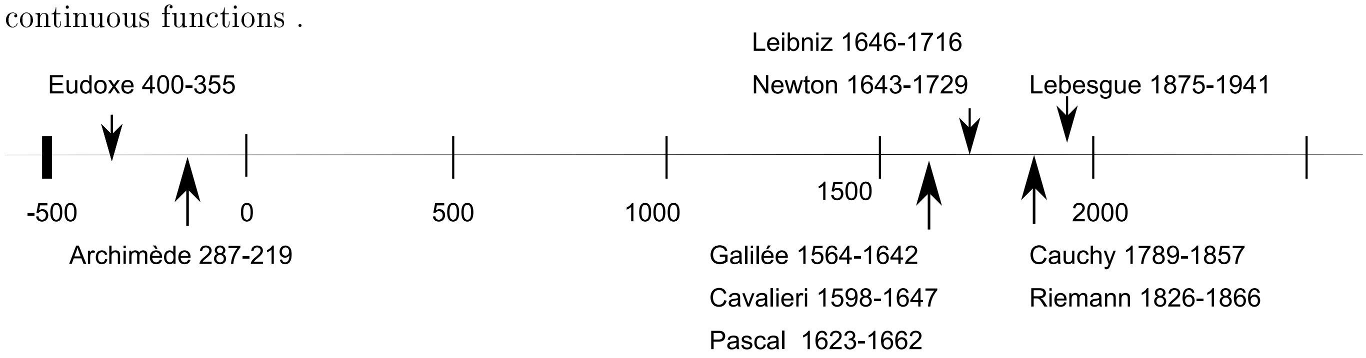
Eudoxus ( 400-355 BC approximately ) first computed areas and volumes using stacking plates whose thickness tends to 0 ; Archimedes ( 287-219 BC) perfects Eudoxe method ( which is mentioned in Euclid's Elements).

At the end of the Middle Ages , (1560-1660) , Cavalieri , Galileo and Pascal enhance the area and volume calculations by stacking small rectangles or parallelepipeds , but not rigorously. However, they get very good approximations.

Newton (1643-1729) and Leibniz (1646-1716) , with the infinitesimal calculus , succeed in proving the relationship between the anti-derivatives of a function and calculus area.( The notation $\int$ is due to Leibniz).

Cauchy (1789-1857) , defines rigorously the concept of limit , and thus gives a rigorous definition of the integral with the continuous functions and Riemann (1826-1866) defines the integral for continuous piecewise .

Lebesgue (1875-1941) extends the concept to classes of more general functions as piecewise continuous functions .



## 1.2 Darboux Sums(1842-1917)

**Definition 1.** *Upper bound, supremum and maximum point.*
Let $f$ be a bounded function on $[a, b]$.

- **An** upper bound of $f$ on $[a, b]$ is a real number $M$ such that for all $x$ of $[a, b]$, $f(x) \leqslant M$.
- **The** supremum of $f$ on $[a, b]$ is the least upper bound, it is is less than any other upper bound. We denote it by $\underset{x\in[a;b]}{sup} f(x)$
- A maximum point of $f$ on $[a, b]$, is a real number $M$ such that for all $x$ of $[a, b]$, $f(x) \leqslant M$ and there exists $x_0 \in [a, b]$ such that $M = f(x_0)$. We denote $\underset{x\in[a;b]}{max} f(x)$

**Example 1.**
True or False ?

1. An upper bound is a maximum point.
2. A maximum point is an upper bound.
3. A bounded function has always a maximum point on $[a, b]$.
4. A bounded function has always a supremum on $[a, b]$.

We define also a lower bound, an infimum and a minimum point.

A partition of an interval $[a, b]$ is a finite sequence of values $x_i$ such that

$\{x_0 = a < x_1 < ... < x_n = b\}$
Each interval $[x_{i-1}, x_i]$ is called a subinterval of the partition. Let $f$ a bounded function on $[a; b]$ and $\sigma = \{x_0 = a < x_1 < ... < x_n = b\}$. be a partition of [a,b].

We set for all $i \in \{1; 2; ...n\}$ :
$m_i = \underset{x\in[x_{i-1};x_i]}{inf} f(x)$, $M_i = \underset{x\in[x_{i-1};x_i]}{sup} f(x)$ and $\delta(\sigma) = \underset{i\in\{1,2,...,n\}}{\max} x_i - x_{i-1}$.

**Definition 2.**
The lower Darboux sum of $f$ with respect to $\sigma$ is :

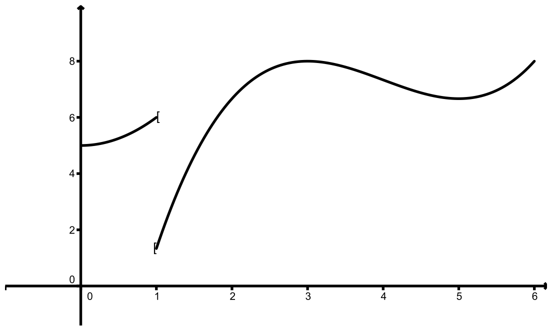
$$s_{[a;b]}(f, \sigma) = \sum_{i=1}^{i=n} m_i(x_i - x_{i-1})$$

.
The upper Darboux sum of $f$ with respect to $\sigma$ is :

$$S_{[a;b]}(f, \sigma) = \sum_{i=1}^{i=n} M_i(x_i - x_{i-1})$$

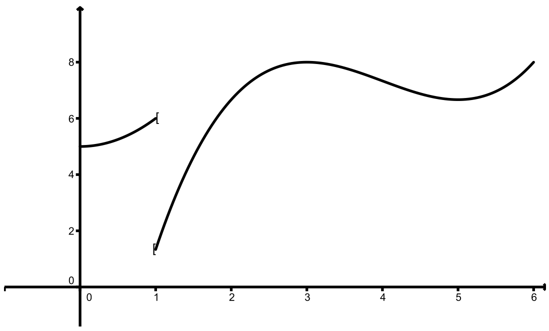
.

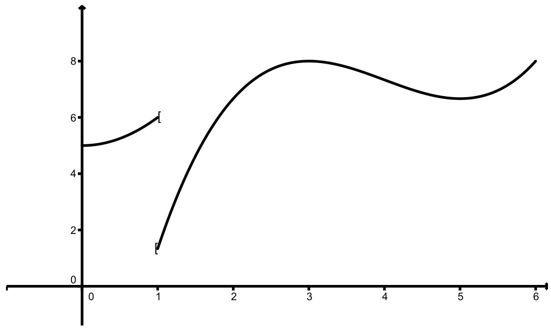
**Example 2.**
Let's consider the function with the following graph :




Let's consider the partition $\sigma$ : $x_0 = a = 0$, $x_1 = 1, 5$, $x_2 = 2$, $x_3 = 4$ and $x_4 = 5 = b$.

1. Justify that $\underset{x\in[x_0;x_1]}{sup} f(x)$ is not a maximum point.
2. Compute Darboux sums.
3. Represent surfaces such that Darboux sums are their areas.

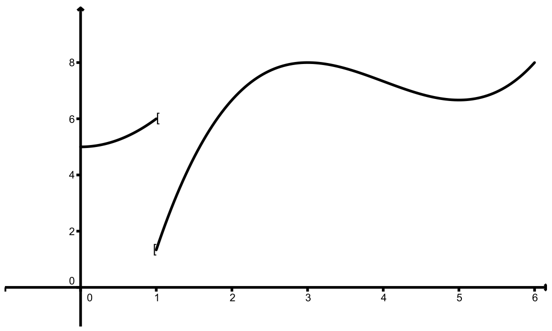





4. Let's consider now the partition $\sigma'$ : $x'_0 = a = 0$, $x'_1 = 1,5$, $x'_2 = 2$, $x'_3 = 3,5$, $x'_4 = 4$ and $x'_5 = 5 = b$.

(a) Check on the graph that :

$$s_{[a;b]}(f,\sigma) < s_{[a;b]}(f,\sigma') < \text{area below the curve} < S_{[a;b]}(f,\sigma') < S_{[a;b]}(f,\sigma)$$




(b) What can you predict on $s_{[a;b]}(f,\sigma)$ and on $S_{[a;b]}(f,\sigma)$ if $\delta(\sigma)$ tends to 0 ?

## 1.3 Riemann Integral

**Definition 3.**
A function $f$ is said Riemann integrable on $[a,b]$ if
$\lim\limits_{\delta(\sigma)\to 0} S(f,\sigma) - s(f,\sigma) = 0$.

The Riemann integral is denoted : $\displaystyle\int_a^b f(x)dx = \lim_{\delta(\sigma)\to 0} S(f,\sigma) = \lim_{\delta(\sigma)\to 0} s(f,\sigma)$.

**Example 3.**
Prove that the function $f$ defined on $[0,1]$ by $f(x) = 1$ if $x \in \mathbb{Q}$ and 0 if not, is not Riemann integrable. (This function is Lebesgue integrable).

**Remark 1.**

- $\int$ is read *sum* as it deals with the limit of $\Sigma$.
- In $f(x)dx$, $f(x)$ matches $M_i$ and $m_i$, $dx$ matches $x_i - x_{i-1}$ as $\delta(\sigma)$ tends to 0.
- If $\sigma$ is a regular subdivision, and $f$ is Riemann integrable on $[a,b]$, we get :
  $\int_a^b f(x)dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(a+k\frac{b-a}{n})\frac{b-a}{n}$.

**Property 1.**
The following functions are Riemann integrable :
- piecewise continuous functions.
- monotonic functions.

## 1.4 Fundamental properties

Let $f$ and $g$ be two Riemann integrable functions on an interval $[a;b]$ and $\lambda$ be a real number.

### 1.4.1 Linearity

$\int_a^b (f+g) = \int_a^b f + \int_a^b g$
$\int_a^b \lambda f = \lambda \int_a^b f$

**Example 4.**

Prove this property

### 1.4.2 Positivity

If $f \geqslant 0$ on the interval $[a;b]$ then $\int_a^b f \geqslant 0$

### 1.4.3 Monotonicity

If $g \geqslant f$ on $[a;b]$ then : $\int_a^b g \geqslant \int_a^b f$.

**Example 5.**

Prove this property

### 1.4.4 Increase in absolute value

$$\left|\int_a^b f\right| \leqslant \int_a^b |f|$$

### 1.4.5 Mean Inequality

$$\left|\int_a^b fg\right| \leqslant \sup|f| \times \int_a^b |g|$$

In particular, (taking g=1) : $\left|\int_a^b f\right| \leqslant \sup|f| \times (b-a)$

### 1.4.6 Mean value of a function

The mean value of $f$ on the interval $[a;b]$ is : $M = \frac{1}{b-a}\int_a^b f$

### 1.4.7 The addition property

$\forall c \in [a;b], \int_a^b f = \int_a^c f + \int_c^b f$

### 1.4.8 Cauchy-Schwarz Inequality

$$\left(\int_a^b fg\right)^2 \leqslant \int_a^b f^2 \times \int_a^b g^2$$

Proofs :

# 2 Antiderivatives

## 2.1 Definition and properties

**Definition 4.**
Let's consider $f : I \to \mathbb{R}$ and $I$ a real interval. $F : I \to \mathbb{R}$ is an antiderivative of $f$ on $I$ if and only if $F$ is differentiable on $I$ and $F' = f$.

**Proposition 1.**
Let's consider $f, F, G : I \to \mathbb{R}$ such that $F$ is an antideriavtive of $f$ on $I$, then $G - F = K$ with $K$ a real constant. Thus antideriavtives of a function only differ from a constant.

## 2.2 Antiderivatives of usual functions

Let $u$ be a function defined on the subset $I$ of $\mathbb{R}$.

| $f(x)$ | $F(x)$ | For$u(x) \in \dots$ |
|---|---|---|
| $u'u^\alpha, \alpha \neq -1$ | $\dfrac{u^{\alpha+1}}{\alpha+1}$ | $\mathbb{R}$ if$\alpha \in \mathbb{N}, \mathbb{R}_+^*$ if $\alpha \in \mathbb{R} - \mathbb{N}$ |
| $\dfrac{u'}{u}$ | $\ln\lvert u\rvert$ | $\mathbb{R}_+^*$ or$\mathbb{R}_-^*$ |
| $u' \cos u$ | $\sin u$ | $\mathbb{R}$ |
| $u' \sin u$ | $-\cos u$ | $\mathbb{R}$ |
| $u' \tan u$ | $-\ln\lvert \cos u\rvert$ | $]-\frac{\pi}{2}; \frac{\pi}{2}[$ |
| $u'e^u$ | $e^u$ | $\mathbb{R}$ |
| $u' \operatorname{ch} u$ | $\operatorname{sh} u$ | $\mathbb{R}$ |
| $u' \operatorname{sh} u$ | $\operatorname{ch} u$ | $\mathbb{R}$ |
| $u' \operatorname{th} x$ | $\ln(\operatorname{ch} u)$ | $\mathbb{R}$ |
| $\dfrac{u'}{1+u^2}$ | $\operatorname{Arctan} u$ | $\mathbb{R}$ |
| $\dfrac{u'}{1-u^2}$ | $\operatorname{Argth} u$ | $]-1; 1[$ |
| $\dfrac{u'}{\sqrt{1-u^2}}$ | $\operatorname{Arcsin} u$ | $]-1; 1[$ |
| $\dfrac{u'}{\sqrt{1+u^2}}$ | $\operatorname{Argsh} u$ | $\mathbb{R}$ |
| $\dfrac{u'}{\sqrt{u^2-1}}$ | $\operatorname{Argch} u$ | $]1; +\infty[$ |
| $\dfrac{u'}{\cos^2(u)}$ | $\tan u$ | $]-\frac{\pi}{2}; \frac{\pi}{2}[$ |
| $\dfrac{u'}{\operatorname{ch}^2 u}$ | $\operatorname{th} u$ | $\mathbb{R}$ |

**Example 6.**

Compute those antiderivatives :

1. $f_1(x) = 2xe^{x^2}$
2. $f_2(x) = \dfrac{\sin(x)}{x}$

3. $f_3(x) = \dfrac{2x}{x^2+1}$
4. $f_4(x) = \dfrac{2}{4x^2+1}$
5. $f_5(x) = \dfrac{1}{x}\ln(x)$

## 2.3 Fundamental theorem of differential calculus

Let $f$ be a continuous function on a real interval $I$ and $a \in I$.The function $F : x \to \int_a^x f(t)dt$ is the unique antiderivative of $f$ which vanishes at $a$.
Thus we get :

$$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$$

**Example 7.** Compute $\int_1^2 (x+1)dx$.

**Remark 2.**

Let $f$ be a continuous function on the interval $I$, the not $\int f(x)dx$ refers to any antiderivative of $f$. Thus, for instance, we get : $\int x^2 dx = \frac{1}{3}x^3 + C$

**Example 8.**

1. Let $f$ be a continuous function on $I$, let's define for $a \in I$ $F(x) = \int_a^x f(t)dt$. Compute $F'$.
2. We assume that $f$ is differentiable on $I$, let's compute $\int_a^b f'(t)dt$

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**Remark 3.** Optional

If $f$ is not a continuous function, the previous theorem is false :

- $f$ could have antiderivatives even though $f$ is non integrable.
  Example :
  Let's consider the function $F$ defined on $[0\,;1]$ by $F(x) = x^2\sin(\frac{1}{x^2})$ sur $]0\,;1]$ et $F(0) = 0$.
  W show that $F$ is differentiable on $[0\,;1]$ and $F'(x) = 2x\sin(\frac{1}{x^2}) - \frac{2}{x}\cos(\frac{1}{x^2})$ sur $]0\,;1]$ and $F'(0) = 0$.
  Let $h$ be the function defined by $h(x) = 2x\sin(\frac{1}{x^2})$ on $]0\,;1]$ with $h(0) = 0$, let $g$ be $g(x) = -\frac{2}{x}\cos(\frac{1}{x^2})$ on $]0\,;1]$ with $g(0) = 0$.
  $h$ is a continuous function on $[0\,;1]$ and so admits an antiderivative $H$ on $[0;1]$.
  Finally we get $g = F' - h = F' - H' = (F-H)'$, so $g$ admits $F - H$ as antiderivative.
  But $g$ is not integrable on $[0\,;1]$, as $g$ is not bounded at 0.
- A piecewise function is integrable but has no antiderivative.

# 3 Change of variable and integration by parts

## 3.1 Integration by parts

Let $u, v : [a; b] \to \mathbb{R}$ be of differentiability class $\mathcal{C}^\infty$ on $[a; b]$. We get :

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v$$

**Example 9.**
Calcuate $\int_0^1 x\sin(2x)dx$

## 3.2 Change of variable

### 3.2.1 General Case

Let $f : I \to \mathbb{R}$, be a continuous function on the interval $I$ and $\phi : [a; b] \to I$, of differentiability class $\mathcal{C}^1$ on $[a; b]$. We get :

$$\int_a^b f(\phi(t))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} f(x)dx$$

In practise, we may use this formula from the left to the to the right, otr from the right to the left, to calulate an antiderivative :

**From the left to the right**

- We set $x = \phi(t)$, and replace $\phi(t)$ by $x$.
- We calculate $dx = \phi'(t)dt$, and replace $\phi'(t)dt$ by $dx$.
- We change the limits of the integral : $t$ varies from a $a$ to $b$ thus $x$ varies from $\phi(a)$ à $\phi(b)$.

**Example 10.**
Calculate : $\int_0^{\frac{\pi}{4}} \frac{dt}{\cos t}$ setting $x = \sin t$

Compute $\int_4^9 \frac{\sqrt{t}}{1+t}$ by setting $x = \sqrt{t}$, then give antiderivatives for $f(x) = \frac{\sqrt{t}}{1+t}$.

**From the right to the left**

- We set $x = \phi(t)$, and replace $x$ by $\phi(t)$.
- We calculate $dx = \phi'(t)dt$, and replace $dx$ by $\phi'(t)dt$.
- We change the limits of the integral : $x$ varies from $\phi(a)$ to $\phi(b)$ thus $t$ varies from $a$ to $b$.

**Example 11.**
Calculate : $\int_0^1 \sqrt{1-x^2}\, dx$ setting $x = \cos t$ and give an antiderivative for $f(x) = \sqrt{1-x^2}$

**To calculate an antiderivative**
We ignore the limits of the integral.

- From the left to the right : we replace $x$ by $\phi(t)$.
- From the right to the left : $\phi$ requires to be a bijection from $I$ to $f(I)$, thus we replace $t$ by $\phi^{-1}(x)$.

**Example 12.**
Calculate antiderivatives in examples 10 et 11.

### 3.2.2 Bioche's rules

Let $f$ be a function defined by $f(t) = \dfrac{P(\cos(t), \sin(t))}{Q(\cos(t), \sin(t))}$ where $P$ and $Q$ are two polynomials functions of two variables, with real coefficients.
In order to calculate $\int f(t)dt$, we define $\omega(t) = f(t)dt$.
We will use the change of variable :

- $u = \cos(t)$, if $\omega(-t) = \omega(t)$.
- $u = \sin(t)$, if $\omega(\pi - t) = \omega(t)$.
- $u = \tan(t)$, if $\omega(\pi + t) = \omega(t)$.
- $u = \tan(t/2)$ for all others cases.

Setting $u = \tan \dfrac{t}{2}$ , we get : $\cos t = \dfrac{1-u^2}{1+u^2}$ ; $\sin t = \dfrac{2u}{1+u^2}$ et $\tan t = \dfrac{2u}{1-u^2}$.

**Example 13.** Find the good change of variables for those integrals :

1. $\displaystyle\int \frac{\cos^2(t)\sin(t)}{1+\cos(t)}dt$
2. $\displaystyle\int \frac{\cos(t)}{1+\sin(t)}dt$
3. $\displaystyle\int \frac{\cos(t)}{\sin(t)(1+\cos^2(t))}dt$
4. $\displaystyle\int \frac{\sin(t)}{1+\sin(t)}dt$

# 4 To calculate antiderivatives

To calculate an antiderivative of $f$, we may use one of the following method :

1. use the inverse of derivatives formula : $f$ is of the form $\dfrac{u'}{u}$, $u'u^n$, etc
2. Integration by parts
   - Classical examples : $f(x) = P(x)e^{ax}$, $f(x) = P(x)\sin(ax)$ and $f(x) = P(x)Ln(Q(x))$ with $P$ and $Q$ two polynomial functions.
   - If $I$ is an antiderivative, then $I$ is solution of a differential equation.
3. Case where $f(x) = \sin^n x \cos^p x$
   - If $n$ and $p$ are even, then we linearize $f$.
   - If $n$ or $p$ is odd, we write $f$ as a sum $u'u^k$ with $u = cos$ or $u = sin$.
4. Antiderivative of a rational function.
   - We use partial fraction decomposition for $f$.
5. Change of variables
   - General Case
   - Bioche's rules
   - Antiderivative of $f(\sqrt{ax+b})$ with $f$ a rational function.

# 5 Application of integral calculus

## 5.1 Area calculus

**Property 2.**
Let $f$ be a continuous function on$[a, b]$.

- If $f$ is **positive** on $[a, b]$ then $\int_a^b f(x)dx$ is the area of the region bounded by the graph of $f$, the $x$-axis and the vertical lines $x = a$ and $x = b$.
- If $f$ is **negative** on $[a, b]$ then $-\int_a^b f(x)dx$ is the area of the region bounded by the graph of $f$, the $x$-axis and the vertical lines $x = a$ and $x = b$.
- If $f$ is of any sign, $\int_a^b f(x)dx = \sum$ areas of regions above the x axis$-\sum$ areas of regions below the x

**Example 14.**
Calculate $\int_0^3 x - 2dx$ and give a geometrical interpretation of this integral.

## 5.2 Center of gravity of a homogeneous plate

Let $S$ be an homogenenous plate with constant thickness and uniform density. The center of gravity is computed tha,ks to a double integral, but in a the particular case where the surface is bounded by the graph of a function $f$, the $x$-axis and the lines of equations $x = a$ and $x = b$, we get the point with coordinates :
$x_G = \frac{1}{A}\int_a^b xf(x)dx$ et $y_G = \frac{1}{2A}\int_a^b (f(x))^2dx$ with $A$ the area of the surface.

**Example 15.**
Calculate the center of inerty of the surface bounded by $y = 2\sqrt{x}$, the $x$-axis and the line $x = h$.

# 6 Exercises

**Exercise 1.**

1. Let's put, for all real number $x$, $I(x) = \int_0^x tdt$.

(a) $I$ is an integral ? an antiderivative ?

(b) Draw it and with an area calculus, find its expression in function of $x$.

(c) Check your result by computing an antiderivative,

(d) by using the formula given in the first remark.

2. Evaluate the following limits using Darboux sums :

$$u_n = \sum_{k=1}^{n} \frac{n+k}{n^2+k^2}$$

$$v_n = \sum_{k=1}^{n} \frac{k}{n^2}\sin(\frac{k\pi}{n})$$

$$w_n = \frac{1}{n}\sqrt[n]{\prod_{k=1}^{n}(n+k)}$$

(You could change the product in sum...)

**Exercise 2.**
Compute the following antiderivatives :

1. $\displaystyle\int \frac{dx}{(2x+1)^3}$
2. $\displaystyle\int \frac{dt}{(1-t)^2}$
3. $\displaystyle\int \frac{du}{\sqrt{1+u}}$
4. $\displaystyle\int \sqrt{1-x}dx$
5. $\displaystyle\int \frac{x^2+1}{\sqrt{x}}dx$
6. $\displaystyle\int \frac{(1-t)^2}{t\sqrt{t}}dt$
7. $\displaystyle\int \frac{z}{\sqrt{z^2-1}}dz$
8. $\displaystyle\int \frac{t}{1+t^2}dt$
9. $\displaystyle\int \frac{t+1}{t^2+4}dt$
10. $\displaystyle\int \frac{x}{(1+x^2)^2}dx$
11. $\displaystyle\int \frac{e^x}{\operatorname{ch} x}dx$
12. $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}}dx$
13. $\displaystyle\int \frac{\sqrt{x}-x^3e^{2x}+x^2}{x^3}dx$

**Exercise 3.**

1. Determine the average value over a period of a purely sinusoidal signal $u(t) = u_0 cos(\omega t+\varphi_0)$
2. Determine the mean value over a period of a triangular wave of period T :
   For $-\frac{T}{2} \leqslant t \leqslant 0, s(t) = -a\left(\frac{4t}{T}+1\right)$ and for $0 \leqslant t \leqslant \frac{T}{2}, s(t) = a\left(\frac{4t}{T}-1\right)$
3. The effective value $u(t)$ is defined as the square root of the average on a period of $u^2(t)$. The effective value is said to be the quadratic average of $u$. Let's determine $u_{eff}$ and $s_{eff}$ for the previous signals (1 and 2).

**Exercise 4.**
Calculate those antiderivatives using an integration by parts

1. $\displaystyle\int x\ln(1+x)\,dx$
2. $\displaystyle\int \operatorname{Arctan}(2x)\,dx$
3. $\displaystyle\int x\operatorname{Arctan} x dx$
4. $\displaystyle\int \operatorname{Arcsin} x dx$
5. $\displaystyle\int \theta\sin 2\theta d\theta$
6. $\displaystyle\int x^2e^{-x}dx$
7. $\displaystyle\int \frac{\alpha}{\cos^2\alpha}d\alpha$
8. $\displaystyle\int x^3\operatorname{Arctan} x dx$

**Exercise 5.**
Compute $\displaystyle\int \sqrt{x^2+1}dx$ using an integration by parts.

**Exercise 6.** Compute this antiderivative(using a double integration by parts) :

$$I(x) = \int_0^x \cos(2t)e^t dt$$

**Exercise 7.**
Calulate using linearization :

1. $\displaystyle\int \cos^2 x dx$ 2. $\displaystyle\int \operatorname{sh}^2 t dt$ 3. $\displaystyle\int \cos^2 x\sin 2x dx$

**Exercise 8.**
Calulate without linearization :
1. $\int \cos^5 x dx$ 2. $\int \text{sh}^3 t dt$ 3. $\int \cos^2 x \sin 2x dx$

**Exercise 9.**
Calculate those integrals (using the given change of variables) :

1. $\int \frac{x^3}{\sqrt{x+1}} dx$ $\left(t = \sqrt{x+1}\right)$ 2. $\int \frac{1+\sqrt{\frac{1+x}{x}}}{x} dx (u = \sqrt{\frac{1+x}{x}})$ 3. $\int \sqrt{a^2 - x^2} dx$ $(x = a \sin t)$
4. $\int \frac{\text{sh}^3 x}{\text{ch}^5 x} dx$ $(y = \text{ch}\, x)$ 5. $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} (u = \sqrt[6]{x})$

**Exercise 10.**
Compute, using Bioche's rules :
1. $\int \tan^4 \theta d\theta$ 2. $\int \frac{1-\cos x}{1+\cos x} dx$ 3. $F(x) = \int \frac{1}{1+\tan x} dx$ 4. $F(t) = \int \frac{1}{\sin t} dt$

**Exercise 11.**
Let's consider an homogeneous plate made by the set of points M(x ;y) whose coordinates check : $0 \leqslant x \leqslant 2$ et $0 \leqslant y \leqslant \frac{x}{x+1}$. Donner les coordonnées du centre de gravité de la plaque.

**Exercise 12.**
A horizontal cylindrical vessel of length l and whose base radius is R, contains a liquid on a height h. Show that the volume V of the liquid is : $V = 2l \int_0^h \sqrt{R^2 - (x-R)^2} dx$
Calculate it using this change of variables : $x - R = R \sin \theta$