Integration and anti-derivatives

Calculus deals principally with two geometric problems :

- (i) The problem of finding SLOPE of the tangent line to the curve, is studied by the limiting process known as differentiation and
- (ii) Problem of finding the AREA of a region under a curve is studied by another limiting process called Integration.

Actually integral calculus was developed into two different directions over a long period independently.

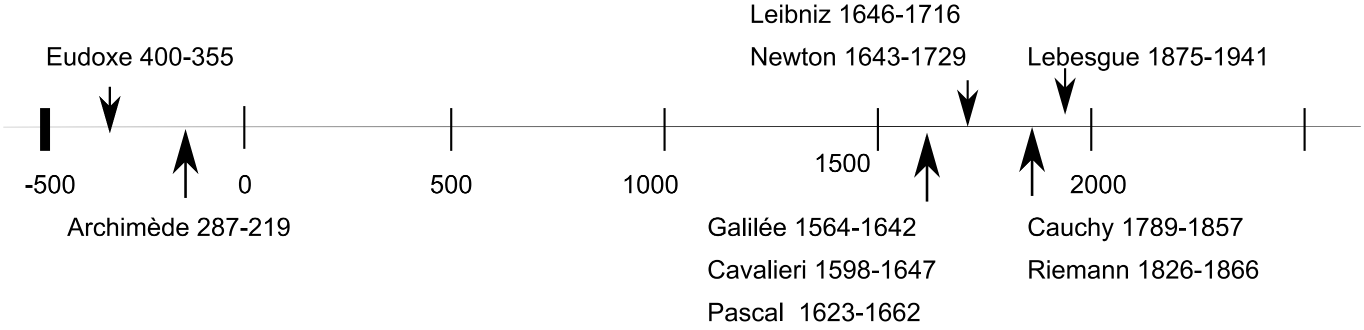
- (i) Leibnitz and his school of thought approached it as the anti derivative of a differentiable function.
- (ii) Archimedes, Eudoxus and others developed it as a numerical value equal to the area under the curve of a function for some interval.

However as far back as the end of the 17th century it became clear that a general method for solution of finding the area under the given curve could be developed in connection with definite problems of integral calculus.

1 Riemann Integral

1.1 Introduction

Eudoxus (400-355 BC approximately) first computed areas and volumes using stacking plates whose thickness tends to 0 ; Archimedes (287-219 BC) perfects Eudoxe method (which is mentioned in Euclid's Elements).
At the end of the Middle Ages , (1560-1660) , Cavalieri , Galileo and Pascal enhance the area and volume calculations by stacking small rectangles or parallelepipeds , but not rigorously. However, they get very good approximations.
Newton (1643-1729) and Leibniz (1646-1716) , with the infinitesimal calculus , succeed in proving the relationship between the anti-derivatives of a function and calculus area.(The notation $\int$ is due to Leibniz).
Cauchy (1789-1857) , defines rigorously the concept of limit , and thus gives a rigorous definition of the integral with the continuous functions and Riemann (1826-1866) defines the integral for continuous piecewise .
Lebesgue (1875-1941) extends the concept to classes of more general functions as piecewise continuous functions .



1.2 Approximation by rectangles

Here, we try to give an approximation of the area under the graph of a function f smooth enough (for example continuous or piecewise continuous) on a segment $[a, b]$.
A partition of an interval $[a, b]$ is a finite sequence of values x_i such that
$\{x_0 = a < x_1 < ... < x_n = b\}$
Each interval $[x_{i-1}, x_i]$ is called a subinterval of the partition. Let f a bounded function on $[a; b]$ and $\sigma = \{x_0 = a < x_1 < ... < x_n = b\}$ be a partition of $[a, b]$. The length of the i-th part is given by $x_i - x_{i-1}$, and we additionnally set $\delta(\sigma) = \max_{i \in \{1,2,...,n\}} x_i - x_{i-1}$.

Example 1. The regular subdivision in n parts of $[a, b]$ is the subdivision such that for all i, $x_i - x_{i-1} = \dfrac{b-a}{n}$, i.e. :

$$\forall i \in \{0, ..., n\}, \quad x_i = a + i\frac{b-a}{n}.$$

To compute the area of f on $[a, b]$, we will approximate the area under its graph on each of the subinterval $[x_{i-1}, x_i]$ of the subdivision. There are many ways to do so (Darboux sums, approximation by rectangles on the right/left, approximation by diamonds). Here, we will introduce the **approximation by rectangles** on the right.

Definition 1. Approximation by rectangles on the right.
The approximation of the area of f by rectangles on the subdivision σ is given by

$$S(f, \sigma) = \sum_{i=1}^{n} f(x_i).(x_i - x_{i-1}).$$

Example 2. We set $f(x) = 1$ and $g(x) = x$ for all $x \in [0, 1]$.

1. Give the expression of the regular subdivision of $[0, 1]$ with n parts.
2. Draw the graphs of f and g and compute their area on $[0, 1]$.
3. We give $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Compute the approximation of the area by rectangles for f and g with the regular subdivision.

1.3 Riemann Integral

Definition 2.

Let f be a continuous (or piecewise continuous) on $[a, b]$. Let (σ_n) be a sequence of subdivisions such that $\delta(\sigma_n) \underset{n\to+\infty}{\to} 0$. Then the sequence of the approximations by rectangles $S(f, \sigma_n)$ converges as n goes to infinity, and we write :

$$\int_a^b f(x)dx = \lim_{n\to+\infty} S(f, \sigma_n).$$

We call the limit the **integral of f on** $[a, b]$.

Remark 1.

- $\displaystyle\int$ is read *sum* as it deals with the limit of Σ.
- In $f(x)dx$, $f(x)$ matches the $f(x_i)$, dx matches $x_i - x_{i-1}$ as $\delta(\sigma)$ tends to 0.
- The limit does not depend of the choice of the subdivisions (σ_n).
- If σ_n is a regular subdivision, we call the approximation by rectangles a **Riemann Sum on** $[a, b]$ and we get :

$$\int_a^b f(x)dx = \lim_{n \to +\infty} \sum_{k=1}^{n} f(a + k\frac{b-a}{n})\frac{b-a}{n}.$$

In particular, if $[a, b] = [0, 1]$, we obtain

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx.$$

Remark 2.
This integral does not make sense for some functions. For example, the function defined on $[0, 1]$ by $f(x) = 1$ if $x \in \mathbb{Q}$ and 0 otherwise is not integrable in this sense (this function will however be integrable for the Lebesgue integral).

1.4 Fundamental properties

Let f and g be two Riemann integrable functions on an interval $[a; b]$ and λ be a real number.

1.4.1 Linearity

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

$$\int_a^b \lambda f = \lambda \int_a^b f$$

1.4.2 Positivity

If $f \geqslant 0$ on the interval $[a; b]$ then $\displaystyle\int_a^b f \geqslant 0$

1.4.3 Monotonicity

If $g \geqslant f$ on $[a; b]$ then : $\displaystyle\int_a^b g \geqslant \int_a^b f$.

1.4.4 Increase in absolute value

$$\left|\int_a^b f\right| \leqslant \int_a^b |f|$$

1.4.5 Mean Inequality

$$\left|\int_a^b fg\right| \leqslant \sup|f| \times \int_a^b |g|$$

In particular, (taking g=1) : $\left|\int_a^b f\right| \leqslant \sup|f| \times (b-a)$

1.4.6 Mean value of a function

The mean value of f on the interval $[a;b]$ is : $M = \frac{1}{b-a}\int_a^b f$

1.4.7 The addition property

$\forall c \in [a;b],\ \int_a^b f = \int_a^c f + \int_c^b f$

1.4.8 Cauchy-Schwarz Inequality

$$\left(\int_a^b fg\right)^2 \leqslant \int_a^b f^2 \times \int_a^b g^2$$

1.5 Area calculus

Property 1.
Let f be a continuous function on$[a,b]$.

- If f is **positive** on $[a,b]$ then $\int_a^b f(x)dx$ is the area of the region bounded by the graph of f, the x-axis and the vertical lines $x=a$ and $x=b$.
- If f is **negative** on $[a,b]$ then $-\int_a^b f(x)dx$ is the area of the region bounded by the graph of f, the x-axis and the vertical lines $x=a$ and $x=b$.
- If f is of any sign, $\int_a^b f(x)dx = \sum$ areas of regions above the x axis$-\sum$ areas of regions below the x

2 Antiderivatives

2.1 Definition and properties

Definition 3.
Let's consider $f: I \to \mathbb{R}$ and I a real interval. $F: I \to \mathbb{R}$ is an antiderivative of f on I if and only if F is differentiable on I and $F' = f$.

Proposition 1.
Let's consider $f, F, G : I \to \mathbb{R}$ such that F is an antideriavtive of f on I, then $G - F = K$ with K a real constant. Thus antideriavtives of a function only differ from a constant.

Example 3. 1. Give an antiderivative of $f(x) = 4x^3$.

2. Give **all** the antiderivatives of the function $g(x) = x^3$.

2.2 Fundamental theorem of differential calculus

Let f be a continuous function on a real interval I and $a \in I$.The function $F : x \to \int_a^x f(t)dt$ is the unique antiderivative of f which vanishes at a.
Thus we get :

$$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$$

Remark 3.

Let f be a continuous function on the interval I, the not $\int f(x)dx$ refers to any antiderivative of f. Thus, for instance, we get : $\int x^2 dx = \frac{1}{3}x^3 + C$

Remark 4. When the function is not continuous, the theorem does not make sense : A piecewise function is integrable but has no antiderivative.

Example 4. Compute $\int_1^3 x^3 dx$.

2.3 Antiderivatives of usual functions

Let u be a function defined on the subset I of $\mathbb{R}$.

$f(x)$	$F(x)$	For $u(x) \in \dots$
$u'u^\alpha, \alpha \neq -1$	$\dfrac{u^{\alpha+1}}{\alpha+1}$	$\mathbb{R}$ if $\alpha \in \mathbb{N}, \mathbb{R}_+^*$ if $\alpha \in \mathbb{R}-\mathbb{N}$
$\dfrac{u'}{u}$	$\ln\lvert u\rvert$	$\mathbb{R}_+^*$ or $\mathbb{R}_-^*$
$u'\cos u$	$\sin u$	$\mathbb{R}$
$u'\sin u$	$-\cos u$	$\mathbb{R}$
$u'\tan u$	$-\ln\lvert\cos u\rvert$	$]-\frac{\pi}{2};\frac{\pi}{2}[$
$u'e^u$	e^u	$\mathbb{R}$
$u'\,\mathrm{ch}\,u$	$\mathrm{sh}\,u$	$\mathbb{R}$
$u'\,\mathrm{sh}\,u$	$\mathrm{ch}\,u$	$\mathbb{R}$
$u'\,\mathrm{th}\,x$	$\ln(\mathrm{ch}\,u)$	$\mathbb{R}$
$\dfrac{u'}{1+u^2}$	$\mathrm{Arctan}\,u$	$\mathbb{R}$
$\dfrac{u'}{1-u^2}$	$\mathrm{Argth}\,u$	$]-1;1[$
$\dfrac{u'}{\sqrt{1-u^2}}$	$\mathrm{Arcsin}\,u$	$]-1;1[$
$\dfrac{u'}{\sqrt{1+u^2}}$	$\mathrm{Argsh}\,u$	$\mathbb{R}$
$\dfrac{u'}{\sqrt{u^2-1}}$	$\mathrm{Argch}\,u$	$]1;+\infty[$
$\dfrac{u'}{\cos^2(u)}$	$\tan u$	$]-\frac{\pi}{2};\frac{\pi}{2}[$
$\dfrac{u'}{\mathrm{ch}^2\,u}$	$\mathrm{th}\,u$	$\mathbb{R}$

Example 5. Compute the following integrals :

1. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{t+1} + \cos(t) + e^t dt$
2. $\displaystyle\int_0^1 2x.e^{x^2} dx$
3. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{1+(\cos(x))^2} dx$

3 Change of variable and integration by parts

3.1 Integration by parts

Let $u, v : [a;b] \to \mathbb{R}$ be of differentiability class $\mathcal{C}^\infty$ on $[a;b]$. We get :

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v$$

Example 6.

Calculate $\displaystyle\int_0^1 x\sin(2x)dx$

3.2 Change of variable

3.2.1 General Case

Let $f : I \to \mathbb{R}$, be a continuous function on the interval I and $\phi : [a; b] \to I$, of differentiability class $\mathcal{C}^1$ on $[a; b]$. We get :

$$\int_a^b f(\phi(t))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} f(x)dx$$

In practise, we may use this formula from the left to the to the right, otr from the right to the left, to calulate an antiderivative :

From the left to the right

- We set $x = \phi(t)$, and replace $\phi(t)$ by x.
- We calculate $dx = \phi'(t)dt$, and replace $\phi'(t)dt$ by dx.
- We change the limits of the integral : t varies from a a to b thus x varies from $\phi(a)$ à $\phi(b)$.

Example 7.

Calculate : $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dt}{\cos t}$ setting $x = \sin t$

Compute $\displaystyle\int_4^9 \frac{\sqrt{t}}{1+t}$ by setting $x = \sqrt{t}$, then give antiderivatives for $f(x) = \dfrac{\sqrt{t}}{1+t}$.

From the right to the left

- We set $x = \phi(t)$, and replace x by $\phi(t)$.
- We calculate $dx = \phi'(t)dt$, and replace dx by $\phi'(t)dt$.
- We change the limits of the integral : x varies from $\phi(a)$ to $\phi(b)$ thus t varies from a to b.

Example 8.

Calculate : $\displaystyle\int_0^1 \sqrt{1-x^2}\, dx$ setting $x = \cos t$ and give an antiderivative for $f(x) = \sqrt{1-x^2}$

To calculate an antiderivative

We ignore the limits of the integral.

- From the left to the right : we replace x by $\phi(t)$.
- From the right to the left : ϕ requires to be a bijection from I to $f(I)$, thus we replace t by $\phi^{-1}(x)$.

Example 9.

Calculate antiderivatives in examples 7 et 8.

3.2.2 Case of trigonometric functions

If $f(x) = \cos(x)^n \sin(x)^p$, we can use trigonometric relations to transform an expression that we don't know how to compute.

- Either using linearization : we recall that $\cos^2 x = \dfrac{1+\cos 2x}{2}$ and $\sin^2 x = \dfrac{1-\cos 2x}{2}$.
- If one of the power is odd, we can utilize the relation $\cos^2 + \sin^2 = 1$:

Example 10.

1. Compute $\int_0^{\pi} (\sin(t))^2 dt$.
2. Compute $\int_0^{\pi} (\cos(x))^3 . \sin(x)^2 dx$.

If f is a function defined by $f(t) = \dfrac{P(\cos(t), \sin(t))}{Q(\cos(t), \sin(t))}$ where P and Q are polynomial functions, one can try to compute the integral using an appropriate change of variables.

Example 11. Using the change of variable $u = \cos(t)$, compute $\int_a^b \dfrac{(\cos(t))^3}{\sin(t)(1 + \cos^2(t))} dt$.

4 To calculate antiderivatives

To calculate an antiderivative of f, we may use one of the following method :

1. use the inverse of derivatives formula : f is of the form $\dfrac{u'}{u}$, $u'u^n$, etc
2. Integration by parts
   - Classical examples : $f(x) = P(x)e^{ax}$, $f(x) = P(x)\sin(ax)$ and $f(x) = P(x)\ln(Q(x))$ with P and Q two polynomial functions.
   - If I is an antiderivative, then I is solution of a differential equation.
3. Case where $f(x) = \sin^n x \cos^p x$
   - If n and p are even, then we linearize f.
   - If n or p is odd, we write f as a sum $u'u^k$ with $u = cos$ or $u = sin$.
4. Antiderivative of a rational function.
   - We use partial fraction decomposition for f.
5. Change of variables

5 Application of integral calculus

5.1 Center of gravity of a homogeneous plate

Let S be an homogenenous plate with constant thickness and uniform density. The center of gravity is computed tha,ks to a double integral, but in a the particular case where the surface is bounded by the graph of a function f, the x-axis and the lines of equations $x = a$ and $x = b$, we get the point with coordinates :
$x_G = \dfrac{1}{A} \int_a^b x f(x) dx$ et $y_G = \dfrac{1}{2A} \int_a^b (f(x))^2 dx$ with A the area of the surface.

Example 12.
Calculate the center of inerty of the surface bounded by $y = 2\sqrt{x}$, the x-axis and the line $x = h$.

6 Exercises

Exercise 1. 1. Without computation, determine the value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(t) dt$.

2. Draw the curve $y = \sqrt{1-x^2}$ for $x \in [-1,1]$ (one can study y^2). Deduce the value of $\int_{-1}^{1} \sqrt{1-x^2}dx$.

Exercise 2. Writing the limits of the following sequences using integrals :

1. $u_n = \sum_{k=1}^{n} \frac{n+k}{n^2+k^2}$
2. $v_n = \sum_{k=1}^{n} \frac{k}{n^2} \sin(\frac{k\pi}{n})$
3. $w_n = \frac{1}{n} \sqrt[n]{\prod_{k=1}^{n}(n+k)}$

   For the last one, we could try to change the product into a sum...

Exercise 3. Compute $\int_0^3 (x-2)dx$ and give a graphical interpretation of it.

Exercise 4.
Compute the following integrals and antiderivatives :

1. $\int_0^2 \frac{dx}{(2x+1)^3}$
2. $\int_2^3 \frac{dt}{(1-t)^2}$
3. $\int_0^1 \frac{du}{\sqrt{1+u}}$
4. $\int_{-1}^0 \sqrt{1-x}dx$
5. $\int_1^2 \frac{x^2+1}{\sqrt{x}}dx$
6. $\int_2^3 \frac{(1-t)^2}{t\sqrt{t}}dt$
7. $\int \frac{z}{\sqrt{z^2-1}}dz$
8. $\int \frac{t}{1+t^2}dt$
9. $\int \frac{t+1}{t^2+4}dt$
10. $\int \frac{x}{(1+x^2)^2}dx$
11. $\int \frac{e^x}{\text{ch}\, x}dx$
12. $\int \frac{x+1}{\sqrt{1-x^2}}dx$
13. $\int \frac{\sqrt{x}-x^3e^{2x}+x^2}{x^3}dx$

Exercise 5.

1. Determine the average value over a period of a purely sinusoidal signal $u(t) = u_0 cos(\omega t+\varphi_0)$
2. The effective value $u_{eff}(t)$ is defined as the square root of the average on a period of $u^2(t)$. The effective value is said to be the quadratic average of u. Let's determine u_{eff} for the previous signal.

Exercise 6.
Calculate those integrals and antiderivatives using an integration by parts

1. $\int_0^1 x \ln(1+x)\, dx$
2. $\int_0^3 \text{Arctan}\,(2x)\, dx$
3. $\int_0^1 x \,\text{Arctan}\, x dx$
4. $\int \text{Arcsin}\, x dx$
5. $\int \theta \sin 2\theta d\theta$
6. $\int x^2 e^{-x} dx$
7. $\int \frac{\alpha}{\cos^2 \alpha} d\alpha$
8. $\int x^3 \,\text{Arctan}\, x dx$

Exercise 7.
Compute $\int \sqrt{x^2+1}dx$ using an integration by parts.

Exercise 8. Compute this antiderivative(using a double integration by parts) :

$$I(x) = \int_0^x \cos(2t)e^t dt$$

Exercise 9.
Calulate using linearization :

1. $\int \cos^2 x dx$ 2. $\int \operatorname{sh}^2 t dt$ 3. $\int \cos^2 x \sin 2x dx$

Exercise 10.
Calulate without linearization :

1. $\int_0^{\pi/2} \cos^5 x dx$ 2. $\int \operatorname{sh}^3 t dt$ 3. $\int \cos^2 x \sin 2x dx$

Exercise 11.

1. i) Find $a, b \in \mathbb{R}$ such that $\frac{1}{(x-1)(x-4)} = \frac{a}{x-1} + \frac{b}{x-4}$.

   ii) Compute $\int_2^3 \frac{1}{t^2 - 5t + 4} dt$.
2. Compute $\int_0^2 \frac{x^3}{x+1} dx$. One could write the numerator as a polynomial expression in $(x+1)$.

Exercise 12.
Calculate those integrals (using the given change of variables) :

1. $\int_{-1}^0 \frac{x^3}{\sqrt{x+1}} dx \quad \left(t = \sqrt{x+1}\right)$
2. $\int_1^2 \frac{1 + \sqrt{\frac{1+x}{x}}}{x} dx$ with $u = \sqrt{\frac{1+x}{x}}$
3. $\int_0^{\frac{a}{2}} \sqrt{a^2 - x^2} dx \quad (x = a \sin t)$
4. $\int \frac{\operatorname{sh}^3 x}{\operatorname{ch}^5 x} dx \quad (y = \operatorname{ch} x)$
5. Compute $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$ with $u = \sqrt[6]{x}$

Exercise 13.
Compute the following integrals using the indicated change of variables :

1. $\int_0^{\frac{\pi}{4}} \tan^4 \theta d\theta$, with $x = \tan(\theta)$.
2. $\int_0^{\frac{\pi}{4}} \frac{1}{1 + \tan x} dx$ with $y = \tan(x)$.
3. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin t} dt$ with $x = \cos(t)$.

Exercise 14.
Let's consider an homogeneous plate made by the set of points M(x ;y) whose coordinates check : $0 \leqslant x \leqslant 2$ et $0 \leqslant y \leqslant \dfrac{x}{x+1}$. Donner les coordonnées du centre de gravité de la plaque.

Exercise 15.
A horizontal cylindrical vessel of length l and whose base radius is R, contains a liquid on a height h. Show that the volume V of the liquid is : $V = 2l\displaystyle\int_0^h \sqrt{R^2 - (x-R)^2}dx$
Calculate it using this change of variables : $x - R = R\sin\theta$