SYSTEMS OF LINEAR EQUATIONS

1 Solutions of linear equations system

Definition 1.
A system of linear equations with n rows and p columns is of the shape : :

$$(S) : \begin{cases} a_{11}x_1 + \cdots + a_{1p}x_p = b_1 \\ \qquad \ddots \\ a_{n1}x_1 + \cdots + a_{np}x_p = b_n \end{cases}$$

- where the variables $(x_1, \ldots, x_p) \in \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$
- Where the coefficients of the system, ie the a_{ij} and the b_j are given. It will be assumed that the coefficients are not all zero.

A solution to a system of linear equations is a set of values for the variables which makes all the equations true .
A system is said homogeneous, if $b_i = 0, \forall i \in \{1, \ldots, n\}$.
A sytem can have

- More equations than variables, it is then said superabundant
- Less equations than variables, it is then said to be sub-abundant
- As many equations as variables, it is said to be square

2 Interpretation with matrices

Matrices are an effective tool for the theoretical resolution of systems and for their explicit resolution. Thus, the system (S) is written in the matrix form as follows : $AX = B$ where $A = (a_{ij})_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant p}} \in \mathcal{M}_{n,p}(\mathbb{K})$ et $B = (b_i)_{1\leqslant i\leqslant n} \in \mathcal{M}_{n,1}(\mathbb{K})$ are given matrices,
$X = (x_j)_{1\leqslant j\leqslant p} \in \mathcal{M}_{p,1}(\mathbb{K})$, matrices of variables.

Exemple 1.

Determine A, B, X, n et p. $(S_2) : \begin{cases} x + y = 1 \\ 2x + y = 3 \\ 3x + 5y = 1 \end{cases}$

3 Cramer's sytems

Definition 2.
A linear system (S) is said **of Cramer** if it satisfies the following two conditions :

(i) The system (S) is square, ie there are as many equations as unknowns, ie $n = p$,

(ii) It admits a unique solution

Proposition 1.
The following properties are equivalent

(i) (S) is a system of Cramer,

(ii) The matrix A of the matrix equation $AX = B$ associated with this system is invertible

(iii) $\det(A) \neq 0$,

(iv) The homogeneous system associated with (S) admits only the zero vector as a solution.

The components of the unique solution of the Cramer system (S) are given for all $j \in \{1, \dots, n\}$ by :

$$x_j = \frac{\det(C_1, \dots, C_{j-1}, B, C_{j+1}, \dots, C_n)}{\det A}$$

where the $C_1, \dots, C_n$ are the column of the matrix A.
In practice, this means that A is replaced in A by B.

Exemple 2.
Solve this system :

$$\begin{cases} x - z = 1 \\ -y - z = 1 \\ -x + 2y = 1 \end{cases}$$

4 Resolution of a linear system in the general case

4.1 Rank method

Definition 3.
The rank of the matrix$A$ associated o this system is called the rank of a linear system (S) and we denote by r the implicit rank of the linear mapping f canonically associated with A.
Thus, $r = \text{rg}(S) = \dim Imf$.
It is thus observed that the rank of a system is entirely determined by the "left part" of the system, ie the part where the variables appear. The right-hand side does not count in the calculation of the rank of the system.

Exemple 3.

Let the following system where m be a real parameter : $\begin{cases} x + y = 1 \\ y + z = 2 \\ x + 2y + mz = 3 \end{cases}$

Find the rank of this system (it depends on m)

To solve a system (S), we define a **square matrix** B **of** A**, of dimension the rank of** S.

4.1.1 Number of variables = rank of S

- The system corresponding to the rows of B is solved. This system is a Cramer's system.
- We consider if the solutions found are solutions of the other equations.

Exemple 4. Solve by the rank method the following system : $(S_2) : \begin{cases} x+y=1 \\ 2x+y=3 \\ 3x+5y=1 \end{cases}$

4.1.2 Number of variables > rank of S

- We write on the left the variables corresponding to the columns of B, and on the right the other variables.
- The system is solved with as variables the variables of the left part, called principal variables. The letters on the right side become parameters.
- It is then necessary to replace the principal unknowns in the possible equations not used in B, and to check that the solutions found with B are also solution of these equations.

If this is the case the set of solutions is a space of dimension $p-r$, otherwise the system has no solution.
Let us illustrate this with examples.

Exemple 5.
The number of rows (n) is less than the number of unknowns (p).
Solve by the rank method the following system :

$$(S_1) : \begin{cases} x+y+z+t=1 \\ x-y-z+t=2 \\ x+y+z-t=3 \end{cases}$$

4.1.3 With one parameter

Exemple 6.
Solve exemple 3.

4.2 Gaussian elimination

We use the same technique as for the search of the rank of a matrix, in order to put the system (S) in the form of a staggered one. Then we discuss the number of solutions through this staggered system. Let us illustrate this with examples.

.

Exemple 7.

1. Solve examples 2,3 et 4 with Gaussian elimination.
2. Solve with Gaussian elimination $(S) : \begin{cases} x + 2y - z = 0 \\ x + y + z = 1 \\ -x + y - 3z = 1 \end{cases}$

5 Theory : Rouche-Fontene's theorem

The set of solutions of a linear system (S) is by definition $S = \{x \in \mathbb{K}^p / f(x) = b\}$ where x is the vector canonically associated to X, and b the matrix associated to B.
This set S is such that :

- $S = \emptyset$ if $b \notin \operatorname{Im} f$
- $S = \{x_0 + v / v \in \operatorname{Ker} f\}$ if $b \in \operatorname{Im} f$ or x_0 is a vector such that $f(x_0) = b$, ie a solution of (S). We recall that $\operatorname{Im} f = \{f(x)/x \in \mathbb{K}\}$.

Thus, it can be seen that there are three possibilities. A system can :

1. have no solutions, it is the case if $b \notin \operatorname{Im} f$ ie that we end up with a system with an absudity
2. have a unique solution, this is the case if : $b \in \operatorname{Im} f$ et $\operatorname{Ker} f = \{O_{\mathbb{K}^p}\}$.
3. have an infinitely many solutions, this is the case if : $b \in \operatorname{Im} f$ and $\operatorname{Ker} f \neq \{O_{\mathbb{K}^p}\}$.

In the case where $b \in \operatorname{Im} f$, we say that the sytem is **compatible**, in the sense that it is compatible with the existence of solutions.

Remarque 1.
$Kerf$ is a vector sub space (a line, plane through the origin...) so if S is not empty,
$S =$ point $+$ vector space, so S is an affine space (Point, line, plane....).

5.1 Bilan

Let (S) be a system of n equations with p unknowns to which we associate the matrix equation $AX = B$. So :

☞ if $n = p$

- ✍ if $\det(A) \neq 0$ then (S) is of Cramer (therefore compatible), it then possesses a unique solution,
- ✍ If $\det(A) = 0$ then the system admits an infinitely many of solutions or no solution.

☞ If $(n \neq p)$, S can have a single solution, no solution or an infinite number of solutions, but, more often than not,

- ✍ If Nbre of equations $<$ Nbre variables,often S has have no solutions
- ✍ If Nbre of equations $>$ Nbre variables, often, S, has no solutions

6 Exercises

Exercice 1. Solve the following systems : m, a, b, c, d are real parameters. λ is a complex parameter. We will give the geometrical interpretation of the different solutions.

$$(S_1) : \begin{cases} x + 2y - z = 0 \\ y - z = 1 \\ -x - y + 3z = 1 \end{cases} \quad (S_2) : \begin{cases} x + z = 1 \\ x + 2y + 3z = 2 \\ x + 4y + 5z = a \end{cases} \quad (S_3) : \begin{cases} 2x - 3y + 2z + 6t - 6v = 3 \\ y + z - 2t = 1 \\ y + z - 2v = 2 \end{cases}$$

$$(S_4) \begin{cases} x - y + z = 4 \\ x + z = 1 \\ x + y = m \\ 2x + y = 1 \end{cases} \quad (S_5) \begin{cases} x + (m+1)y + 2mt = 0 \\ mx + z + t = 1 \end{cases} \quad (S_6*) \begin{cases} x + 2y - z = 0 \\ 2x - y + mz = -1 \\ x + y - 2z = 2 \end{cases}$$

$$(S_7 - facultatif) \begin{cases} x + y + z = 1 \\ x + ay + bz = c \\ x + by + az = c \end{cases}$$