DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

1 Rappel

Exemple 1.
The next two questions are independent.

1. Let (E) be the differential equation : $(y-1)y' = 4x + 2$.
   (a) What can we say about the solutions of (E) for $x = -\frac{1}{2}$?
   (b) Is the f function defined by $f(x) = 2x + 2$ a solution of (E) ?
   (c) A student states that $x \longrightarrow 1 + \sqrt{4x^2 + 4x + k}$ where k is a constant, are solutions of the equation (E)."
   Another student replied : "It is not possible because :
   — We already have a solution.
   — The product of $(y-1)y'$ is the $4x+2$ polynomial, so there can not be a square root in the solution.
   — When the function is replaced in $y'(1-y)$ there will remain the constant k, while in $4x+2$ there is no constant.
   Which of the two students is right ?
   (d) Can there be other solutions ?
2. Determine the solutions of equation $2x' + 3x = t$.

2 Linear differential equations with non-constant coefficients

2.1 First order homogeneous linear equations

Théorème 1.
Let a be a continuous function on I. The solutions of the homogeneous equation (H) : $y' + a(x)y = 0$ are the functions of the form $x \mapsto ke^{-A(x)}$ où $k \in \mathbb{R}$ is any constant and A a primitive of a on I.

Exemple 2.

1. Solve the differential equation : $2y' + xy = 0$.
2. Demonstration of the theorem : Let A be a primitive of a and assume that (H) admits a solution f. Let g be the function defined by $g(x) = f(x)e^{A}$.
   (a) Express f as a function of g.
   (b) Knowing that f is a solution of (H), determine g' and deduce g.
   (c) Deduce the possible solutions of (H).
   (d) Deduce the solutions of (H).

2.2 First order linear equations with second member

According to the course on differential equations seen at the beginning of the year, we know that the solutions are the sum of a particular solution and the general solution of the homogeneous equation. So we have the following proposal :

Proposition 1.
Let a and b be two continuous functions, and A a primitive of a over an interval I. Let (E) be the differential equation : $(E) : y' + a(x)y = b(x)$.

Then the general solution of (E) on I is of the form : $y = g(x) + ke^{-A(x)}$ where $k \in \mathbb{R}$ is any constant and G a particular solution that can be calculated using the "constant variation" method described below.

Constant Variation Method
- We look for g in the form $g(x) = k(x)e^{-A}$.
- We replace g in the equation.
- From this we deduce k'.
- Then k.
- And finally g.

Exemple 3.
Solve on $I =]1; +\infty[$, the differential equation $(E) : y' + \frac{1}{x \ln x} y = \frac{e^x}{\ln x}$

2.3 Equation with initial condition - Cauchy problem

Let $x_0 \in I$ and $y_0 \in \mathbb{R}$. o solve the first-order linear differential equation (E) with the initial condition $y(x_0) = y_0$, determine the solutions y of (E) **unique** (E). The datum of the equation with an initial condition is also called the first-order Cauchy problem.

Exemple 4.
Determine the solution f of the equation $xy' + 2y = x + 1$ on $\mathbb{R}^{+*}$ checking $f(1) = 2$.

3 Equations with separable variables

Définition 1.
A first-order differential equation with separable variables is an equation that can be written as :

$$y'g(y) = f(x)$$

where y is the unknown function of the equation and G and h two continuous functions over an interval I.

Proposition 2.
The equation $y'g(y) = f(x)$ is solved by integrating the two members of the equation and we obtain $G(y) = F(x)$ with G and F of Primitives of the functions g and f respectively.

Exemple 5.
Solve the following differential equations :

1. $y'y^2 = 0$.
2. $y' = y^2$.
3. $y'(e^y + 1) = x$.

Remarque 1.
From the previous example :
— We can sometimes reduce the interval of study to obtain $y'g(y) = f(x)$ (In 2., we take an interval I where : $\forall x \in I, y(x) \neq 0$), the initial equation is therefore not always equivalent to : $y'g(y) = f(x)$.
— We have to study the cases which have been discarded to obtain $y'g(y) = f(x)$ (In 2. we have added the null function as a solution).
— When $G(y) = F(x)$ is obtained, the solution y is implicitly obtained, but we can not always express y as a function of x, (as in 3. for example).

Remarque 2. In physics
In physics, the presentation of the computation is often the following : $\frac{dy}{dx}g(y) = f(x)$ (1) therefore $g(y)dy = f(x)dx$ (2), Then we integrate the two members of equality with respect to y and x.

1. In equality (1), is y a function or a variable ?
2. In equality (2), is y a function or a variable ?
3. Explain this apparent paradox.

Remarque 3. The following demonstration can be found in books or on the internet to solve $y' + a(x)y = 0$:

$$\begin{aligned} y' + ay = 0 \quad &\Leftrightarrow \quad \frac{dy}{dx} = -ay \\ &\Leftrightarrow \quad \frac{dy}{y} = -a\,dx \\ &\Leftrightarrow \quad \ln y = -\int a\,dx + K \quad \Leftrightarrow \quad y = ke^{-\int a\,dx} \end{aligned}$$

1. What is the name of this method ?
2. Are these equivalences correct ?

As we have seen, it is difficult to solve a differential equation rigorously by separating the variables, for the linear differential equations, we will apply directly the results of the paragraph on the linear equations

4 Homogeneous equation

Définition 2.
A differential equation is said to be homogeneous if there exists an interval I on which the equation can be written in the form $y' = g(\frac{y}{x})$.

Exemple 6.
Show that the equation $xy' = x + y$ is a homogeneous equation.

Remarque 4.
Be careful not to confuse : "homogeneous equation" and "homogeneous equation of a linear equation".

Propriété 1.
Let $y' = g(\ fracyx)$ be a homogeneous equation. By putting $z = \ fracyx$, the equation becomes an equation with separable variables in z.

Exemple 7.
Show previous property.

Exemple 8.
Solve the equation : $xy' = x + 2y$.

5 Exercices

Exercice 1. Determine the general solution of the following differential equations :

1. $(1 - x^2)y' = (2 - x)y$, with $y(-2) = 2$.
2. $xy^2y' = x^3 + y^3$
3. $(1 + x^2)y' + 2xy = \frac{1}{x}$ sur $\mathbb{R}_+^*$
4. $x - y + xy' = 0$ sur $]0, +\infty[$
5. $xy' + 3y = \frac{1}{1 - x^2}$ sur $]0; 1[$
6. $(x^2 - y^2)y' = xy$
7. $y' - xy = xe^{x^2}$ sur $\mathbb{R}$
8. $y' \sin x + y \cos x = \sin^2 x$ sur $]0; \Pi[$
9. $y' - xe^{-y} = 0$, $y(1) = 0$.
10. $y^2 + (x + 1)y' = 0$, $y(0) = 1$.
11. $y' = 2x\sqrt{1 - y^2}$
12. $yy' = x$, $y(0) = 4$.
13. $y = \ln y'$, $y(0) = \ln(2)$.

Exercice 2. We consider the first order linear differential equation (E), defined on $I =]0; +\infty[$ by

$$(E) : x(1 + \ln^2(x))y' + 2\ln(x)y = 1$$

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1. For all $x \in I$, we define $f(x) = 1 + \ln^2(x)$. Determine for all $x \in I$, $f'(x)$.
2. Determine the solutions y_H of the homogeneous equation (H) associated with (E).
3. Determine a particular solution y_0 of the differential equation (E).

4. Deduce the solutions of the differential equation (E).
5. What is the unique solution of (E) satisfying the initial condition : $y(1) = 2$.

Exercice 3.
Find a curve of the plane of the equation $y = f(x)$, passing through the point $(0, 3)$ and the slope of the tangent to the point $M(x; y)$ is $\dfrac{2x}{y^2}$.

Exercice 4.
Let be the chemical reaction : $A + B \longrightarrow C$

At time $t = 0$ we have a moles of the A body and b moles of the B body. By putting $x(t)$ the number of moles of C present at time t, we assume that the velocity of occurrence of C follows the law $\dfrac{dx}{dt} = K(a - x)(b - x)$ where k is a positive constant.

1. Solve the differential equation by distinguishing the cases where $a = b$ and $a \neq b$.
2. Represent graphic representations of solutions and verify that they are consistent with experience.
3. Show that when a tends to b, one finds the solution where $a = b$.

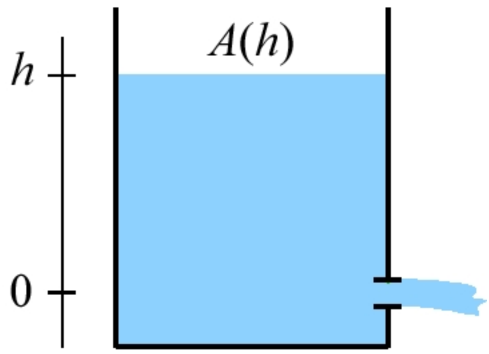
Exercice 5.

The Torricelli law gives a relation between the velocity of a liquid flowing through the orifice of a container and the height of liquid above the orifice.

Let $h(t)$ be the height of liquid contained in the container above the orifice at time t and $A(h)$ the area of the surface of the liquid when the height of the liquid is h. We have the relation (E) :

$$A(h)\frac{dh}{dt} = -k\sqrt{h}$$

Where k is a positive constant depending on factors such as the viscosity of the liquid and the cross-sectional area of the flow hole.



1. Physically justify the sign of $h'(t)$.
2. At time $t = 0$, a cylindrical tank of radius R in meters contains H meters of water above the orifice.
   (a) What guess can we make about the graphical representation of h ?
   (b) Determine the solutions of the differential equation (E) and deduce $h(t)$.
   (c) Graph h. Is the curve consistent with the conjecture of question 1 ?
3. Resume the previous question with a conical reservoir of height H, of radius R and whose orifice is at the summit of the cone.

Exercice 6.
In a perfect gas, which models the behavior of gases at the usual pressures and temperatures, we obtain that the pressure in a gas in equilibrium (without motion) under the effect of gravity obeys the differential equation (E) Next, the variable being the altitude z :

$$\frac{dp}{dz} + \frac{Mg}{RT}p = 0$$

with M : Molar mass of gas, constant ; For air, $M =$ 0,00290 kg/mol $g = 9,81 m/s^2$: gravity acceleration R=8,314 J/K/mol : Perfect gas constant

T represents the temperature of the gas, in kelvins. However, in the atmosphere, the temperature is not constant. A very frequently used model is to postulate a linear decrease in temperature with altitude : $T(z) = T_0(1 - kz)$, T_0 being the temperature at sea level, and the coefficient k being a constant.

Solve the differential equation (E) and represent its solutions.

Exercice 7.
We consider the differential equation : $x(x-1)y' - y(y-1) = 0$

1. Without solving the equation, show that the intersection between the set of integral curves and the line of equation $x = 0$ reduces to two maximum fixed points, and similarly with the line of equation $x = 1$.
2. Solve the Differential Equation $x(x-1)y' - y(y-1) = 0$.
3. Specify the answer to question 1.
4. Represent a few integral curves