MATRICES

Objectives

— **Transpose a matrix.**

— **Compute determinants.**

— **Compute the inverse of a matrix.**

Throughout the chapter we will denote by $\mathbb{K}$ the sets $\mathbb{R}$ or $\mathbb{C}$.

1 Matrices

1.1 Transposition of matrices

Definition 1.
Let $A = (a_{ij})_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant p}} \in \mathcal{M}_{n,p}(\mathbb{K})$ be any matrix. We call transpose of the matrix A, the matrix denoted ${}^tA$ where A^T bleonging to $\mathcal{M}_{p,n}(\mathbb{K})$ and is defined by :

$$ {}^tA = (a_{ji})_{\substack{1\leqslant j\leqslant p\\ 1\leqslant i\leqslant n}} $$

In other words : the lines of ${}^tA$ are the columns of A.

That is to say in extended version :

$$ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{np} \end{pmatrix} \Rightarrow {}^tA = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{1p} & a_{2p} & \cdots & a_{np} \end{pmatrix} $$

Example 1.
Determine ${}^tA$ with

$$ A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R}) $$

Property 1.

(i)

$$ \forall A \in \mathcal{M}_{n,p}(\mathbb{K}), \forall B \in \mathcal{M}_{n,p}(\mathbb{K}), {}^t(A+B) = {}^tA + {}^tB $$

(ii)

$$ \forall A \in \mathcal{M}_{n,p}(\mathbb{K}), \forall \lambda \in \mathbb{K}, {}^t(\lambda A) = \lambda {}^tA $$

(iii)

$$ \forall A \in \mathcal{M}_{n,p}(\mathbb{K}), {}^t\left({}^tA\right) = A $$

(iv)

$$ \forall A \in \mathcal{M}_{n,p}(\mathbb{K}), \forall B \in \mathcal{M}_{p,q}(\mathbb{K}), {}^t(AB) = {}^tB\,{}^tA $$

2 Square matrices

2.1 Particularities of square matrices

Definition 2.
A square matrix is a matrix with the same number of rows and columns. We write $\mathcal{M}_n(\mathbb{K})$, the set of square matrices with coefficients in $\mathbb{K}$, rather than $\mathcal{M}_{n,n}(\mathbb{K})$.

All operations and properties defined on any matrix belonging to $\mathcal{M}_{n,p}(\mathbb{K})$ remain valid on square matrices. There exists, however, a particular matrix called identity matrix, a neutral element for the multiplication of matrices :

Definition 3.
The identity matrix is denoted by I_n and we have :

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

It is also a matrix for which we have commutativity of the matrix product that is to say :

$$\forall A \in \mathcal{M}_n(\mathbb{K}), AI_n = I_nA = A$$

Definition 4.
We can also define the nth power of a square matrix by induction, that is to say :

$$\begin{aligned} &A^0 = I_n \\ &A^n = AA^{n-1} = A^{n-1}A \end{aligned}$$

Theorem 1. Newton's binomial for the matrices
Let X and Y belong to $\mathcal{M}_n(\mathbb{K})$ such that X and Y commute, i.e $XY = YX$ then we have :

$$\forall n \in \mathbb{N}, (X+Y)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} X^{n-k}Y^k$$

Caution, this formula is obviously false if the matrices do not commute.

Example 2.
Let A and B be two matrices that commmute. Develop $(A+B)^2$.

There are other special square matrices which are used quite often :

Definition 5 (The diagonal matrices)**.**
A diagonal matrix is a matrix with all the elements out of the main diagonal [1] null and those of the diagonal are any. (They can therefore be null ! The matrix O is of course a diagonal matrix)

1. ie the diagonal starting from the top to the left to finish in bottom to the right

Example 3.
Are the following diagonal matrices ?

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \text{ et } B = \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & \lambda_2 & 0 \\ \lambda_3 & 0 & 0 \end{pmatrix}$$

Property 2.

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \Rightarrow D^n = \begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^n \end{pmatrix}$$

Remember this property, it will serve us much !

Definition 6 (Triangular matrices)**.**
A matrix *upper triangular* is a square matrix having these coefficients above the main diagonal, diagonal included, not zero, the others being null.

A matrix *lower triangular* is of course the opposite, ie it is a square matrix having these coefficients below the main diagonal, diagonal included, not zero, the others being null.

Example 4.
Write a lower triangular matrix and a upper triangular matrix.

Definition 7 (Trace of a square matrix)**.**
We call trace of $A \in \mathcal{M}_n(\mathbb{K})$, The sum of the elements of the main diagonaland it is denoted $tr(A)$, which means that

$$tr(A) = \sum_{i=1}^{i=n} a_{ii}$$

Example 5.
$A = \begin{pmatrix} 1 & 7 & 3 \\ -3 & 5 & 2 \\ 5 & 3 & -12 \end{pmatrix}$. Calculate $tr(A)$.

Property 3.
Let $A \in \mathcal{M}_n(\mathbb{K})$, be any square matrix and ${}^tA$ it's transpose matrix, then

$$tr(A) = tr\left({}^tA\right)$$

2.2 Determinant of square matrices

The determinant of a matrix can be calculated using the following two definitions :

Definition 8. *Matrice of dimension 2*

let be $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ The determinant of A is denoted $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Definition 9. *Matrice of dimension n with $n \geqslant 3$*
Let $A = (a_{ij})$ be a square matrix of order n, and let A_{ij} the matrix without the i^{th} ligne without j^{th} column of A.

For all $j \in \{1;2...n\}$, $\det A = \sum_{i=1}^{i=n}(-1)^{i+j}a_{ij}\det A_{ij}$ (We develop according to the j^{th} column).

For all $i \in \{1;2...n\}$, $\det A = \sum_{j=1}^{j=n}(-1)^{i+j}a_{ij}\det A_{ij}$ (We develop according to the i^{th} ligne).

Remark 1.
The previous definition is valid for some dimension of the matrix, but there is another way to calculate the determinant of a 3-dimensional matrix, using the so-called Sarrus rule (French mathematician 1798-1861) :

In order to calculate $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$ We copy the first two columns to the right : $\begin{bmatrix} a & d & g & a & d \\ b & e & h & b & e \\ c & f & i & c & f \end{bmatrix}$

Then the sum of the products of the three secondary diagonals is subtracted from the sum of the products of the three main diagonals, and we obtain :

$\det A = aei + dhc + gbf - (ceg + fha + ibd)$.

Example 6.
Calculate the following determinant by expanding in a row and then following a column.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix}$$

2.2.1 Important properties

- $\forall n \geqslant 2, \det I_n = 1$.
- $\forall A, B \in \mathcal{M}_n(\mathbb{K}), \det AB = \det A \det B$
- $\det A = \det {}^tA$
- If the rows or the columns are not linearly independant (for example two columns colinear),then $det(A) = 0$.
- Adding to a column a multiple of a different column does not change the determinant. Same for rows.
- For $\alpha \in \mathbb{R}$, $det(\alpha A) = \alpha^n \det(A)$.

2.3 Inverse of a square matrix

Definition 10.
Let $A \in \mathcal{M}_n(\mathbb{K})$ be a square matrix. We say that the matrix A is invertible if there exists a matrix denoted by $A^{-1} \in \mathcal{M}_n(\mathbb{K})$ such that $AA^{-1} = A^{-1}A = I_n$. We call A^{-1}, the inverse matrix of the matrix A.

Example 7.
Let $A \in M_3(\mathbb{K})$ such that $A^3 + 3A^2 = I_3$

Show that A is invertible and determine it's inverse.

Proposition 2.

- A matrix A is invertible if and only if $\det A \neq 0$.
- If A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$

Example 8.
Demonstrate the previous property.

Proposition 3.
Let $A = (a_{ij})$ be an invertible matrix. We can then compute A^{-1} by :

$A^{-1} = \dfrac{1}{\det\ A}\,{}^{t}B$ with $B = (b_{ij})$ the matrix defined by $b_{ij} = (-1)^{i+j}\det\ A_{ij}$.

Alternatively, we can solve the linear system corresponding to A : $AX = Y \iff X = A^{-1}Y$.

Example 9.

1. Calculate the inverse matrix of : $A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 3 & 1 \\ 0 & 2 & 2 \end{pmatrix}$
2. Verify the preceding property by filling the gaps :

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \ldots$$

$A^{-1} = \dfrac{1}{\det(A)}\,{}^{t}B$ with

$$B = \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} \ldots & \ldots \\ \ldots & \ldots \end{vmatrix} & \begin{vmatrix} \ldots & \ldots \\ \ldots & \ldots \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

$${}^{t}B = \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} \ldots & \ldots \\ \ldots & \ldots \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} \ldots & \ldots \\ \ldots & \ldots \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

$$A^tB = \begin{pmatrix} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \dots & \dots \\ \dots & \dots & \dots \\ a_{31}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{33}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \dots & \dots \end{pmatrix}$$

Therefore,

$$\frac{1}{\det(A)}A^tB = \begin{vmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

Property 4.
Let $A \in \mathcal{M}_n(\mathbb{K})$ be a matrix and $B \in \mathcal{M}_n(\mathbb{K})$ both assumed invertible then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Example 10.
Demonstrate the previous property.

Exercices TD 1-2

Exercise 1.
Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Express ${}^tA$.

Exercise 2. It is proposed to study the nth power of $M = \begin{pmatrix} a & -a \\ -b & b \end{pmatrix}$ where a and b are real numbers.

We suppose that M^n can be written : $M^n = \begin{pmatrix} u_n & -u_n \\ -v_n & v_n \end{pmatrix}$.

1. Write u_{n+1} as a function of u_n, a and b; then v_{n+1} as a function of v_n, a and b.
2. Deduce the expression of M^n as function of n, a and b.
3. Numerical application : $M = \begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}$

Exercise 3.
Let (a_n), (b_n) and (c_n) be three real sequences such as $a_0 = 1$, $b_0 = 2$, $c_0 = 7$, and checking the recurrence relations : $\begin{cases} a_{n+1} = 3a_n + b_n \\ b_{n+1} = 3b_n + c_n \\ c_{n+1} = 3c_n \end{cases}$

We want to express a_n, b_n, and c_n only as a function of n.

1. Consider the column vector $X_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$.

   Find a matrix A such that $X_{n+1} = AX_n$. Deduce $X_n = A^n X_0$.
2. Let $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Calculate N^2, N^3, then N^p for $p \geqslant 3$.
3. Show that : $A^n = 3^n I + 3^{n-1} n N + 3^{n-2} \dfrac{n(n-1)}{2} N^2$.
4. Deduce a_n, b_n, and c_n function of n.

Exercise 4.

1. Check on square matrices of order 2 that Tr AB = Tr BA. (We can show that this result is true for all matrices A and B such as AB and BA exist).
2. Show that the trace is a linear form on the set of square matrices of dimension n.

Exercise 5.
Calculate, if possible, the inverse matrix of each of the following matrices :

1. $A = \begin{pmatrix} 2 & 12 \\ 0,5 & 3 \end{pmatrix}$.
2. $B = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 5 & 4 \\ 1 & -5 & -4 \end{pmatrix}$.
3. $C = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 4 & 0 & -1 \end{pmatrix}$.

Exercise 6.
Calculate the determinant of the following matrices

1. $A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 4 & 2 & 1 & 1 \\ 3 & 6 & 1 & 0 & 1 \\ 4 & 8 & 0 & -1 & 1 \\ 5 & 10 & 0 & 0 & 1 \end{pmatrix}$.
2. $B = \begin{pmatrix} \lambda_1 & 1 & \cdots & 1 \\ 0 & \lambda_2 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$

Exercise 7.
Let M be the matrix $M = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 5 & 4 \\ 1 & -5 & -4 \end{pmatrix}$.

We admit that $M^3 + 2M^2 - M - 2I_3 = O$.
Show that M is invertible and compute its inverse.

Exercise 8.
We say that two matrices A and B are similar if there exists a matrix P invertible such that $A = P^{-1}BP$.

1. Show that two similar matrices have the same determinant.
2. Show in two different ways that if A is invertible then B is also invertible.