Sequence

Objectives

- **Know general concepts and definitions**
- **Compute sequence limits.**
- **Know general theorems on limits.**
- **Be able to study recursive sequences.**

Link to example file in pdf : sequence example.pdf

Link to example file in pdf : sequence exercises.pdf

1 Definitions

1.1 Sequences

Definition 1.
A **sequence** of real numbers is a collection $(u_n)_{n\in\mathbb{N}}$ of real numbers indexed in $\mathbb{N}$. We say that u_n is the **member or term** of the sequence $(u_n)_{n\in\mathbb{N}}$. In other words, to give a real sequence is to give an application : $\begin{array}{l}\mathbb{N} \to \mathbb{R} \\ n \mapsto u_n\end{array}$

Remark 1.
For convenience, the general term u_n is sometimes defined only from a certain rank n_0. We then write the sequence $(u_n)_{n\geqslant n_0}$. Throughout this chapter, everything that is defined from 0 is easy to transpose for n_0.

Definition 2.
Let P be a property relating to the real sequences. We say that a $(u_n)_{n\in\mathbb{N}}$ satisfies P **from a certain rank** if and only if there is a natural number n_0 such that the $(u_n)_{n\geqslant n_0}$ satisfies the P property.

The sequences studied in this chapter are defined in two different ways :

1. Each term is defined from the preceding terms.
   Example : for $n \geqslant 0$, $u_{n+1} = \sqrt{u_n + 2}$ and $u_0 = 1$
   In paragraph 5, the follow-up to $u_{n+1} = f(u_n)$.
2. Each term is defined from its rank.
   Examples : $u_n = n^2 + \ln(n+1)$ or $u_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$

Example 1. Compute the first 3 terms of those sequences.

Video : Correction example 1

1.2 Reasoning by induction

Let n be an integer and $P(n)$ a property. If :

- Initialization or base case : There is an integer n_0 such that $P(n_0)$ is true.
- Inheritance or inductive step : for every integer $m \geqslant n_0$, $P(m)$ true implies that $P(m+1)$ is true.

Then $P(n)$ is true for all $n \geqslant n_0$.

Example 2. Show by induction that : $\displaystyle\sum_{k=1}^{k=n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Video : Correction example 2

1.3 Arithmetic sequences and geometric sequences.

1.3.1 Arithmetic sequences

Definition 3. Let $r \in \mathbb{R}$. An **arithmetic** sequence of common difference r is a sequence $(u_n)_{n\in\mathbb{N}}$ given by the first term u_p and the inductive formula :

$$\forall n \in \mathbb{N}, u_{n+1} = u_n + r$$

We then have :

$$\forall n \in \mathbb{N}, u_n = u_p + (n-p)r$$

Property 1. Sum of consecutive terms
The sum of the consecutive terms of an arithmetic sequence is equal to

$$\textit{number of terms}\ \frac{\textit{1st term + last term}}{2}$$

In particular : $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + ... + n = \frac{n(n+1)}{2}$

1.3.2 Geometric sequences

Definition 4. Let $q \in \mathbb{R}^*$. A **geometric** sequence of common ratio q is a sequence $(u_n)_{n\in\mathbb{N}}$ given by the first term u_p and the inductive formula :

$$\forall n \in \mathbb{N}, u_{n+1} = qu_n$$

We then have :

$$\forall n \in \mathbb{N}, u_n = u_p q^{n-p}$$

Property 2. Sum of consecutive terms
The sum of the consecutive terms of a geometric sequence whose common ratio is different from 1 is equal to

$$\textit{1st term}\ \frac{1 - commonratio^{\textit{number of terms}}}{1 - commonratio}$$

In particular : if $q \neq 1$, $\displaystyle\sum_{k=0}^{n} q^k = 1 + q + ... + q^n = \frac{1-q^{n+1}}{1-q}$

1.4 bounded sequences

Definition 5.
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

1. The sequence $(u_n)_{n\in\mathbb{N}}$ is said to be bounded from **above** if and only if there exist a real number M so that :
$$\forall n \in \mathbb{N}, u_n \leqslant M$$
2. The sequence $(u_n)_{n\in\mathbb{N}}$ is said to be bounded from **below** if and only if there exist a real number m so that :
$$\forall n \in \mathbb{N}, u_n \geqslant m$$
3. The sequence $(u_n)_{n\in\mathbb{N}}$ is said to be **bounded** if it is bounded from above and below

Example 3. Are the following sequences bounded ?

1. $u_n = \dfrac{1}{n}, n > 0$
2. $u_n = \dfrac{n^2+1}{n+3}$
3. $u_n = (-1)^n \cos(n)$.

Video : Correction example 3

Remark 2.
The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded if and only if $(|u_n|)_{n\in\mathbb{N}}$ is bounded from above ie there exist $M \geqslant 0$ so that :
$$\forall n \in \mathbb{N}, |u_n| \leqslant M$$
This widely used property also has the merit of being able to be used with complex sequences.

2 Variations

Definition 6.
Let $(u_n)_{n\in\mathbb{N}}$ a real sequence.

1. We say that $(u_n)_{n\in\mathbb{N}}$ is **increasing** (respectively **strictly increasing**) from the rank n_0 if and only if we have : $\forall n \geqslant n_0, u_n \leqslant u_{n+1}$ (respectively $\forall n \geqslant n_0, u_n < u_{n+1}$)
2. We say that $(u_n)_{n\in\mathbb{N}}$ is **decreasing** (respectively **strictly decreasing**) from the rank n_0 if and only if we have : $\forall n \geqslant n_0, u_n \geqslant u_{n+1}$ (respectively $\forall n \geqslant n_0, u_n > u_{n+1}$)
3. We say that $(u_n)_{n\in\mathbb{N}}$ is **monotonic** (respectively **strictement monotonic**) from the rank n_0 if and only if it is decreasing or increasing (respectively strictly decreasing or increasing)from the rank n_0.
4. We say that $(u_n)_{n\in\mathbb{N}}$ is **constant** if and only if we have : $\forall n \in \mathbb{N}, u_n = u_{n+1}$. We say that $(u_n)_{n\in\mathbb{N}}$ is **stationary** if and only if it is constant starting from a certain rank.

Property 3.
When $(u_n)_{n\in\mathbb{N}}$ is a sequence with strictly positive terms starint a certain rank n_0, It may be useful to use the following equivalent form of the definition :

- $(u_n)_{n\in\mathbb{N}}$ Is increasing from rank $n_0 \Leftrightarrow \forall n \in \mathbb{N},\ n \geqslant n_0,\ \dfrac{u_{n+1}}{u_n} \geqslant 1$
- $(u_n)_{n\in\mathbb{N}}$ Is decreasing from rank $n_0 \Leftrightarrow \forall n \in \mathbb{N},\ n \geqslant n_0,\ \dfrac{u_{n+1}}{u_n} \leqslant 1$

Property 4.
Let (u_n) be a sequence defined by $u_n = f(n)$ and let n_0 be an integer. If f is increasing (respectively decreasing) on $[n_0; +\infty[$ then (u_n) is increasing (respectively decreasing) starting rank n_0.

Method to study variations of a sequence
According to the definition, and the previous property, there are therefore 3 methods :

- Study the sign of $u_{n+1} - u_n$
  (This method can be applied for all sequences, but it is sometimes simpler to use one of the two methods below).
- Compare the ratio $\dfrac{u_{n+1}}{u_n}$ to 1.
  (This method can only be used for **strictly positive** sequences from a certain rank. It must therefore be specified when using this method).
- Study the variations of f, when the sign of f' is easily determined.
  (This method can only be used for the sequences of the form $u_n = f(n)$.)

Example 4.
Study the monotony of the following sequences :

1. $\forall n \in \mathbb{N}, u_n = n^2 - 10n + 21$
2. $\forall n \in \mathbb{N}^*, u_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$
3. $\forall n \in \mathbb{N}, u_n = \dfrac{2^n}{n+1}$

Video : Correction example 4

Property 5. Monotony of geometric sequences
Let (u_n) be a geometric sequence of common ratio q and of first term u_0.
Variations of (u_n) :

	$q<0$	$0<q<1$	$q=1$	$1<q$
$u_0<0$	not monotonic	increasing	constant	decreasing
$u_0>0$	not monotonic	decreasing	constant	increasing

3 Convergence of a sequence

3.1 Finite Limit of a sequence

Definition 7.
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence, $l \in \mathbb{R}$. We say that the sequence $(u_n)_{n\in\mathbb{N}}$ **converges** to the value l and we denote : $\lim\limits_{n\to+\infty} u_n = l$ ou $u_n \underset{n\to+\infty}{\longrightarrow} l$ If and only if :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N, |u_n - l| \leqslant \varepsilon$$

Example 5.
Give an exemple of a sequence that converges to e.

Video : Correction example 5

Remark 3.
The way N depends on ε characterizes what is called the **convergence speed** of the sequence. For ε fixed, plus N is close to 0, the faster the rate of convergence.

Definition 8.
Let x_0 and a be real numbers. We say that a is an approximate value of x_0 up to ε if :
$|x_0 - a| \leqslant \varepsilon$.

Remark 4.
Therefore $u_n \underset{n\to+\infty}{\longrightarrow} l \Leftrightarrow$ for all $\varepsilon > 0$, u_n give an approximation of l starting a certain rank and to the precision ε.

The numerical series are therefore used to find approximate real values whose exact values can not be calculated.

Property 6.
If $u_n \underset{n\to+\infty}{\longrightarrow} l$ then $|u_n| \underset{n\to+\infty}{\longrightarrow} |l|$

Proposition 1. Any convergent sequence is bounded.

Video : optional proof

3.2 Infinite limit of a sequence

Definition 9.
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence. We say that $(u_n)_{n\in\mathbb{N}}$ converges to infinity $+\infty$ (respectively $-\infty$) and we denote $u_n \underset{n\to+\infty}{\longrightarrow} +\infty$ (respectively $u_n \underset{n\to+\infty}{\longrightarrow} -\infty$) if and only if :

$$\forall A \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \geqslant N, u_n \geqslant A$$

(respectively $\forall A \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \geqslant N, u_n \leqslant A$).

Definition 10.
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence. We say that $(u_n)_{n\in\mathbb{N}}$ is **divergent** if it is not convergent.

Example 6.
Give two examples of divergent sequences.

Video : Correction example 7

3.3 Properties on limits

3.3.1 Limits and operations

All limit tables for functions can be reused here.

3.3.2 Limits and inequality

Theorem 2. To the limit
Let $(u_n)_{n\in\mathbb{N}}$ and $(u'_n)_{n\in\mathbb{N}}$ be two real sequences converging to l and l'. We suppose that there exist a certain rank n_0 from which $\forall n \geqslant n_0, u_n \leqslant u'_n$, then $l \leqslant l'$.

Example 7.
Find two (u_n) and (v_n) sequences such that $u_n < v_n$ from a rank n_0 and such that (u_n) and (v_n) converge to the same limit$L$.

Video : Correction example 8

Theorem 3. Squeeze theorem
Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$ be real sequences and l a real number. We suppose that $u_n \underset{n\to+\infty}{\to} l$ and $w_n \underset{n\to+\infty}{\to} l$ and that starting a rank n_0 we have $\forall n \geqslant n_0, u_n \leqslant v_n \leqslant w_n$ then $v_n \underset{n\to+\infty}{\to} l$

Theorem 4. Comparing sequences

1. Let $(u_n)_{n\in\mathbb{N}}$ et $(v_n)_{n\in\mathbb{N}}$ be real sequences. We suppose that : $u_n \underset{n\to+\infty}{\to} +\infty$ and that starting a rank n_0 we have $\forall n \geqslant n_0, u_n \leqslant v_n$ then $v_n \underset{n\to+\infty}{\to} +\infty$
2. Let $(u_n)_{n\in\mathbb{N}}$ et $(v_n)_{n\in\mathbb{N}}$ be real sequences. We suppose that : $v_n \underset{n\to+\infty}{\to} -\infty$ and that starting a rank n_0 we have $\forall n \geqslant n_0, u_n \leqslant v_n$ then $u_n \underset{n\to+\infty}{\to} -\infty$

Example 8.

— Show that $(u_n)_{n\in\mathbb{N}^*}$ defined by : $\forall n \geqslant 1, u_n = 1 + \dfrac{(-1)^n}{n^2+1}$ converges to 1.

— Show that $n! \underset{n\to+\infty}{\to} +\infty$

Video : Correction example 9

3.3.3 Limit of a sequence and continuous functions

Theorem 5.
If f is continuous in a, and if $\lim\limits_{n\to+\infty} u_n = a$, then $\lim\limits_{n\to+\infty} f(u_n) = f(a)$.

Example 9.
Determine the limit of (u_n) defined by $u_n = e^{\frac{(-1)^n}{n}}$

Video : Correction example 10

3.3.4 Limits of geometric sequences

Property 7.
Let $(u_n)_{n\in\mathbb{N}}$ be a geometric sequence of common ratio $q \neq 0$ with $u_0 \neq 0$.

1. if $|q| < 1$ then $u_n \underset{n\to+\infty}{\to} 0$

2. if $q = 1$ then $u_n \underset{n\to+\infty}{\to} u_0$ (the sequence is constant)
3. if $q > 1$ then $u_n \underset{n\to+\infty}{\to} \begin{cases} +\infty \, si \, u_0 > 0 \\ -\infty \; si \; u_0 < 0 \end{cases}$
4. if $q \leqslant -1$ then (u_n) as no limits.

3.3.5 Convergence of monotonic sequences

Theorem 6. 1. Let $(u_n)_{n\in\mathbb{N}}$ be an increasing sequence starting n_0. If it is bounded from above then : $\lim\limits_{n\to+\infty} u_n = \sup\{u_n | n \in \mathbb{N}, n \geqslant n_0\}$
If not bounded from above then the sequence converges to $+\infty$

2. Let $(u_n)_{n\in\mathbb{N}}$ be an decreasing sequence starting n_0. If it is bounded from below then : $\lim\limits_{n\to+\infty} u_n = \inf\{u_n | n \in \mathbb{N}, n \geqslant n_0\}$
If not bounded from below then the sequence converges to $-\infty$

Example 10.
Let (u_n) be a sequence defined by $u_n = \displaystyle\sum_{k=1}^{k=n} \frac{1}{k^2}$. We admit that (u_n) converges to $\dfrac{\pi^2}{6}$.

1. Justify that $u_n \leqslant \dfrac{\pi^2}{6}$ for all $n \geqslant 1$.
2. Let (v_n) be defined by $v_n = \displaystyle\sum_{k=1}^{k=n} \frac{1}{k^3}$. Study the convergence of (v_n).

Video : Correction example 11

3.3.6 Adjacent sequences

Definition 11.
Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences. These suites are said to be **adjacent** if and only if the following two conditions are satisfied :

1. They are monotonous and of opposite variations.
2. $u_n - v_n \underset{n\to+\infty}{\to} 0$

Theorem 7. Convergence
Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences. Then they are convergent and have the same limit. Moreover, their common limit is included for any n between u_n and v_n.

Example 11.
For all $n \geqslant 1$, we have : $u_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}$ and $v_n = u_n + \dfrac{1}{n.n!}$.
Show that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent sequences.
Their common limit is the number e. We will demonstrate this next year.

Video : Correction example 12

3.3.7 Subsequences

Definition 12.
(v_n) is a subsequence of (u_n) if there is a strictly increasing mapping φ from $\mathbb{N}$ to $\mathbb{N}$ so taht :
$\forall n \in \mathbb{N} \qquad v_n = u_{\varphi(n)}$

In practice, we will use the sequences extracted from even and odd indices, that is to say the subsequences : $(u_{2n})_{n\in\mathbb{N}}$ et $(u_{2n+1})_{n\in\mathbb{N}}$

Theorem 8.
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence. We supose that $u_{2n} \underset{n\to+\infty}{\to} l$ et $u_{2n+1} \underset{n\to+\infty}{\to} l$ then $u_n \underset{n\to+\infty}{\to} l$

Example 12.
Study the convergence of $(u_n)_{n\in\mathbb{N}}$ defined by : $u_n = (-1)^n \sin\left(\frac{1}{n}\right)$

Video : Correction example 13

Theorem 9.
All subsequence of a convergent sequence converges to the same limit.

Remark 5.
We can use the contraposition of the previous theorem to prove that a (u_n) sequence is not convergent :

- Show a non convergent subsequence.
- or show two subsequences converging to different limits

Example 13.
Give the nature of (u_n) defined by : $u_n = (-1)^n$.

Video : Correction example 14

4 Linear recursive sequences of order 2 (optional)

Theorem 10. Linear recursive series of order 2
Let's consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by it's first two terms u_0 and u_1 and the relation :

$$\forall n \in \mathbb{N}, u_{n+2} = au_{n+1} + bu_n$$

a and b being real numbers.

We call **caracteristic equation**, the equation $(EC) : x^2 - ax - b = 0$. Let α and β be the solutions of this equation then :

- If α and β are real numbers ie $\Delta = a^2 + 4b > 0$, then there exist two real numbers A and B so that :
$$\forall n \in \mathbb{N}, u_n = A\alpha^n + B\beta^n$$
- if α and β are eaual ie $\Delta = a^2 + 4b = 0$, then there exist two real numbers A and B so that :
$$\forall n \in \mathbb{N}, u_n = (An + B)\alpha^n$$

— If α and β are complex conjugate, ie $\Delta = a^2 + 4b < 0$, they can be written : $\alpha = \rho e^{i\theta}$ and $\beta = \rho e^{-i\theta}$. then there exist two real numbers A and B so that :

$$\forall n \in \mathbb{N}, u_n = \rho^n \left(A\cos(n\theta) + B\sin(n\theta)\right)$$

In each case, the constants A and B are obtained by solving a system of two equations with two unknowns as a function of u_0 and u_1.

Example 14.
The fibonacci sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_0 = u_1 = 1$ and

$$\forall n \in \mathbb{N}, u_{n+2} = u_{n+1} + u_n$$

Give an expression of u_n as a function of n.

Video : Correction example 15

5 Recursive real sequences $u_{n+1} = f(u_n)$

5.1 Definition

Example 15.

1. Represent the function defined by $f(x) = 2\sqrt{2x-4}$ for $x \in [2; 6]$, and the ligne of equation $y = x$.
2. Discuss the existence of the sequence (u_n) defined by $u_{n+1} = f(u_n)$, using the graph, depending on the value of u_0.
3. Let $I = [4; +\infty[$. Show that if $u_0 \in I$ then the sequence (u_n) is defined.

Video : Correction example 16

Property 8.
Let f be a real function defined on an interval I of $\mathbb{R}$. Let $a \in I$. The recursive sequence (u_n) defined by $u_0 = a$ and $\forall n \in \mathbb{N}, \quad u_{n+1} = f(u_n)$ is defined if $f(I) \subset I$, and then $u_n \in I$ for all $n \in \mathbb{N}$.

Throughout the remainder of the paragraph, we consider :

— I an interval of $\mathbb{R}$.

— f a function so that $f(I) \subset I$.

— A sequence (u_n) defined by $u_{n+1} = f(u_n)$ and $u_0 \in I$.

5.2 Variations

Property 9.
If f is increasing on I, then (u_n) is monotonic :

— increasing if $f(u_0) \geqslant u_0$ i.e. if $u_1 \geqslant u_0$.

— decreasing if $f(u_0) \leqslant u_0$ i.e. if $u_1 \leqslant u_0$.

Example 16.
Prove the preceding theorem.

Video : Correction example 17

5.3 Convergence

Property 10.
If I is bounded and if f is increasing, then for all $u_0 \in I$, the sequence (u_n) is convergent.

Example 17.
Prove the preceding theorem.

Video : Correction example 18

Property 11.
If f is continuous on I and if (u_n) converges to $l \in I$, then $f(l) = l$ i.e. l is an attractive fixed point f.

Example 18.
Prove the preceding theorem.

Video : Correction example 19

Example 19.
Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined in 15, ie $u_0 = 6$ and $\forall n \in \mathbb{N}, u_{n+1} = f(u_n)$ with $f(x) = 2\sqrt{2x-4}$.

1. Justifiy the sequence $(u_n)_{n\in\mathbb{N}}$ is well defined.
2. Study the variations of (u_n).
3. Using the different properties above, show that (u_n) converges and determine its limit.

Video : Correction example 20

Remark 6. Example : Let us look for a solution of the equation $f(x) = 0$ with $f(x) = x^4 + 3x + 1$.

We show that f is strictly increasing and continuous on [-1 ; 0], that $f(-1) = -1$ and $f(0) = 1$. According to the intermediate value theorem, there exists a unique real $x_0\ in]-1;0[$ such that $f(x_0) = 0$.

We can also show that the following two sequences converge to x_0 :

— Method of Lagrange : $u_{n+1} = \dfrac{-1}{u_n^3 + 3}$ et $u_0 = -1$

— Method of Newton : $v_{n+1} = \dfrac{3v_n - 1}{4v_n^3 + 3}$ et $v_0 = 0$

To get an approximate value of x_0 to 10^{-15}, just calculate v_4, then go to u_{12}. Newton's method is, in general, a method which converges much faster than the Lagrange method.

It is found that -0.337666765642802 is an approximate value of x_0 to 10^{-15} near.

6 Comparaison of sequences

6.0.1 Negligeable sequences

Definition 13.
Let $(u_n)_{n\in\mathbb{N}}$ et $(v_n)_{n\in\mathbb{N}}$ be two real sequences.We say that $(u_n)_{n\in\mathbb{N}}$ is negligeable in front of $(v_n)_{n\in\mathbb{N}}$ when $n \to +\infty$ if and only if :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N, |u_n| \leqslant \varepsilon |v_n|$$

We denote : $u_n \underset{n\to+\infty}{=} o\,(v_n)$ or We also say that u_n is infinitely small with respect to v_n and that v_n is infinitely large with respect to u_n when $n \to +\infty$.

Property 12.
If (v_n) is non-zero from a certain rank :

$u_n = o(v_n) \Leftrightarrow \dfrac{u_n}{v_n} \underset{n\to+\infty}{\to} 0$

Example 20.

1. Show that $n! = o(n^n)$.
2. Is the sequence $(\dfrac{1}{n})$ negligeable in front of $(\dfrac{1}{n^2})$?
3. Is the sequence $(1, 1^n)$ negligeable in front of (n^{1000}) ?

Video : Correction example 21

6.0.2 Equivalent sequences

6.0.3 Definition

Definition 14.
Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be real sequences We say that $(u_n)_{n\in\mathbb{N}}$ is equivalent to $(v_n)_{n\in\mathbb{N}}$ when $n \to +\infty$ if and only if $u_n - v_n = o(v_n)$. We denote $u_n \underset{+\infty}{\sim} v_n$ or $u_n \sim v_n$.

We also say that u_n is an equivalent of v_n.

Thus, a simple characterization of equivalence when the terms of the $(v_n)_{n\in\mathbb{N}}$ are non-zero from a certain rank is :

$$u_n \underset{+\infty}{\sim} v_n \Leftrightarrow \frac{u_n}{v_n} \underset{n\to+\infty}{\to} 1$$

Remark 7. Careful
Do not confuse properties $u_n \underset{+\infty}{\sim} v_n$ and $u_n - v_n \to 0$. There is no relationship of implication between them. Indeed :

- Take $u_n = \dfrac{1}{n}$ and $v_n = \dfrac{1}{n^2}$ then $u_n - v_n \underset{n\to+\infty}{\to} 0$ but $\dfrac{u_n}{v_n} \underset{n\to+\infty}{\to} +\infty \neq 1 \Rightarrow u_n \underset{+\infty}{\not\sim} v_n$
- Take $u_n = n^2 + n$ and $v_n = n^2$ then $\dfrac{u_n}{v_n} = 1 + \dfrac{1}{n} \underset{n\to+\infty}{\to} 1 \Rightarrow u_n \sim v_n$. yet $u_n - v_n = n$ does not cenverge to 0 !

6.0.4 Research of equivalents

For a (u_n) defined by $u_n = f(n)$, it is possible, where possible, to find an equivalent of u_n using a limited development of f in $+\infty$.

Example 21.

Give an equivalent to $u_n = n \ln\left(\sqrt{\dfrac{n+1}{n-1}}\right)$

Video : Correction example 22

7 Exercises

Exercise 1.

Determine the variations and limits of the following sequences :

1. $u_n = e^n - 5n$
2. $u_{n+1} = -5u_n$ and $u_0 = 3$.
3. $u_{n+1} = 0,5u_n$ and $u_0 = -3$
4. $u_{n+1} = -0,5u_n$ and $u_0 = -5$.

Exercise 2.

Let (u_n), (v_n) and (w_n) be real sequences verifying :

For all integer $n > 0$: $u_n \leqslant v_n \leqslant w_n$.

Check the affirmations that are true and justify your choices.

1. If the sequence (v_n) converges to $-\infty$, then :
   - ☐ the sequence (w_n) converges to $-\infty$.
   - ☐ the sequence (u_n) is bounded from above .
   - ☐ the sequence (u_n) converges to $-\infty$.
   - ☐ the sequence (w_n) has no limits.
2. If $u_n \geqslant 1$, $w_n = 2u_n$ and $\lim u_n = l \in \mathbb{R}$ then
   - ☐ $\lim v_n = l$.
   - ☐ the sequence (w_n) converges to $+\infty$.
   - ☐ $\lim\ (w_n - u_n) = l$.
   - ☐ We can not say whether the sequence (v_n) has a limit or not.
3. If lim u_n = -2 and lim w_n = 2, then :
   - ☐ the sequence (v_n) is bounded from above
   - ☐ lim (v_n) =0
   - ☐ the sequence (v_n) has no limits
   - ☐ We can not say whether the sequence (v_n) has a limit or not.
4. Si $u_n = \dfrac{2n^2-1}{n^2}$ et $w_n = \dfrac{2n^2+3}{n^2}$ alors :
   - ☐ lim (w_n)=0

- ☐ $\lim\ (v_n) = 2$
- ☐ $\lim\ (u_n) = 2$
- ☐ the sequence (v_n) has no limits

Exercise 3.
Calculate the sum of the first n odd numbers.

Exercise 4.
We call arithmetic-geometric sequence, any sequence $(u_n)_{n\in\mathbb{N}}$ defined by its first term u_0 and by a relation of type : $\forall n \in \mathbb{N}, u_{n+1} = au_n + b$ where a and b are real numbers.

1. Express u_n as a function of n when $a = 1$.
2. Suppose $a \neq 1$
   (a) Verify that the equation $\ell = a\ell + b$ admits a unique solution l.
   (b) We set $w_n = u_n - \ell$. Verify that $(w_n)_{n\in\mathbb{N}}$ is a geometric sequence.
   (c) Express u_n as a function of a, b, n, u_0.
   (d) Determine the value of $\displaystyle\sum_{k=0}^{n} u_k$ as a function of a, b, n, u_0.

Exercise 5.
Conjecture the u_n value as a function of n for the following $(u_n)_{n\in\mathbb{N}}$ and then prove by induction your conjectures $u_0 \in \mathbb{R}$ unless otherwise stated :

1. $u_1 = 5$ and $\forall n \in \mathbb{N}^*, u_{n+1} = \dfrac{n}{2}u_n$
2. $\forall n \in \mathbb{N}, u_{n+1} = e^{n-2}u_n$
3. $u_1 = 1$ and $\forall n \in \mathbb{N}^*, u_{n+1} = -\dfrac{n}{n+1}u_n$
4. $\forall n \in \mathbb{N}, u_{n+1} = 2u_n + 2^n$
5. $\forall n \in \mathbb{N}, u_{n+1} = (u_n)^2$

Exercise 6.
Study the convergence and monotony of (v_n) defined by : $v_n = \cos\left(n\dfrac{\pi}{4}\right)$.

Exercise 7.

1. Show that for all k > 0 : $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$
2. Deduce that $u_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$ is convergent.
3. we set for $n \in \mathbb{N}^*, v_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$
   — Show that $v_n \leqslant 1 + u_{n-1}$
   — Deduce that the sequence (v_n) is convergent.

Exercise 8.
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence defined by it's first term $u_0 = 2$ and the inductive relation : $\forall n \in \mathbb{N}, u_{n+1} = \dfrac{1}{2}\left(u_n + \dfrac{3}{u_n}\right)$

1. Show that $(u_n)_{n\in\mathbb{N}}$ exists.
2. Show that $(u_n)_{n\in\mathbb{N}}$ is bounded from below by $\sqrt{3}$
3. Study the variations of $(u_n)_{n\in\mathbb{N}}$ then show that it converges to $\sqrt{3}$

Exercise 9. Let $(u_n)_{n\in\mathbb{N}}$ be defined by $u_0 = 1$ and : $\forall n \in \mathbb{N}, u_{n+1} = \frac{1}{4}u_n^2 + 1$

1. Show that the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing.
2. Prove by induction that $(u_n)_{n\in\mathbb{N}}$ is bounded from above. Conclude.
3. Determine $\lim\limits_{n\to+\infty} u_n$.
4. What happens if we take $u_0 = 3$?

Exercise 10.
We set $S_n = \sum\limits_{k=1}^{n} \frac{(-1)^k}{k}$, $u_n = S_{2n}, v_n = S_{2n+1}$.
Show that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent.
Deduce the convergence of the sequence $(S_n)_{n\in\mathbb{N}}$

Exercise 11.
Give an equivalent of the sequences below as $\frac{k}{n^\alpha}$:

1. $u_n = (1 + \frac{a}{n})^n$
2. $u_n = \frac{n^2+3}{n^5+1}$
3. $u_n = \frac{\sin(n) + n}{\sqrt{n} + \cos(n)}$
4. Is the following statement true or false : $\frac{\ln(n)}{n^3} \sim \frac{1}{n^3}$ because $\ln(n)$ is negligeable in front of n^3.

Exercise 12.

1. Show that $u_n = \frac{e^{-n}+2}{n^2+1}$ is negligeable in front of $\frac{k}{n^\alpha}$ and determine k and α.
2. (a) Do we have $\frac{\sin^2(n)}{n^2} = o(\frac{1}{n^3})$?

   (b) and $\frac{\sin^2(n)}{n^2} = o(\frac{1}{n^2})$?
3. Show that $\frac{\ln(n)}{n^2}) = o(\frac{1}{n^{\frac{3}{2}}})$

Exercise 13.
It is proposed to study the evolution of a population of ladybugs using a model using the numerical function f defined by $f(x) = kx(1-x)$, k being A parameter that depends on the environment ($k \in \mathbb{R}^{+*}$). In the chosen model, it is assumed that the number of ladybugs remains less than one million. The number of ladybugs, expressed in millions of individuals, is

approximated for the year n by a real number u_n, with u_n between 0 and 1. For example, if for year zero There are 300 000 ladybugs, one will take $u_0 = 0.3$. It is assumed that the evolution from one year to the next obeys the relation $u_{n+1} = f(u_n)$, f being the function defined above. The aim of the exercise is to study the behavior of the sequence (u_n) for different values of the initial population u_0 and the parameter k. We will study the variations of f and the sign of $f(u0) - u0$ by distinguishing the cases where $k \in]0, 1]$ and $k \in [1, 2]$.