Taylor expansions

Objectifs
- **Know common Taylor expansions.**
- **Calculate taylor expansions by different techniques.**
- **Know when to apply taylor expansions.**

Throughout this chapter, I represents any interval of $\mathbb{R}$. $\mathcal{F}(I, \mathbb{R})$ represents the set of functions defined from I to $\mathbb{R}$.

1 Little o notation

Definition 1.
Let I be a real interval and a a real. $a \in \mathbb{R}$ or a is an endpoint of I. Let f and g be two functions of $\mathcal{F}(I, \mathbb{R})$. f is a little "o" of g at the neighborhood of a where $a \in [-\infty, +\infty]$, if and only if :

1. Case $a \in \mathbb{R}$: $\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in I, |x-a| \leqslant \alpha \Rightarrow |f(x)| \leqslant \varepsilon |g(x)|$
2. Case $a = +\infty$: $\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in I, x \geqslant A \Rightarrow |f(x)| \leqslant \varepsilon |g(x)|$
3. Case $a = -\infty$: $\forall \varepsilon > 0, \exists A \in \mathbb{R}, \forall x \in I, x \leqslant A \Rightarrow |f(x)| \leqslant \varepsilon |g(x)|$

$f(x) \underset{x\to a}{=} o\left(g(x)\right)$
we write $f(x) \underset{x\to a}{=} o\left(g(x)\right)$ (f is little-o of g) or if there is no confusion $f = o(g)$. We also say that $f(x)$ is infinitely small with respect to $g(x)$ at the neighborhood of a.

Proposition 1 (Characterization)**.**
The following sentences are equivalent :

1. $f(x) \underset{x\to a}{=} o\left(g(x)\right)$
2. If $g \neq 0$ at the neighborhood a, $\dfrac{f(x)}{g(x)} \underset{x\to a}{\to} 0$
3. There exists a function ε such that $f(x) = g(x)\varepsilon(x)$ avec $\varepsilon(x) \underset{x\to a}{\to} 0$ at neighborhood of a.

Example 1.

1. Find all natural numbers n such that $\dfrac{x^3}{1+x^2} = o(x^n)$ at the neighborhood of 0.
2. Let f be a function such that $f(x) = o(x^3)$ at the neighborhood of 0. Find natural numbers n such that $\dfrac{f(x)}{x} = o(x^n)$.

Video : example 1

We get properties for "o", like the comparative growth theorem :
en $+\infty$: $x^\alpha = o(x^\beta)$ ssi $\alpha < \beta$, $x^\alpha = o(e^x)$, $\ln x = o(x^\beta)$
en 0^+ : $x^\beta = o(x^\alpha)$ ssi $\alpha < \beta$, $\ln x = o\left(\frac{1}{x^\alpha}\right)$

2 Taylor expansion

In the following, n denotes a integers and $a \in \mathbb{R}$

2.1 Taylor expansion at 0

Definition 2.
Let I be a real interval such that $0 \in I^o$, $f : I \to \mathbb{R}$, $n \in \mathbb{N}$. f has a serie expansion truncated of at order n at the neighborhood of 0, denoted by $DL_n(0)$ if and only if there exists a real polynomial P_n of degree less or equal than n, such that :

$$f(x) - P_n(x) = o(x^n)$$

at the neighborhood of 0.
A $DL_n(0)$ of f is written :

$$f(x) = P_n(x) + o(x^n)$$

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$$

.

Remark 1.
Whatever is the situation, the little $o()$ is an "abstract" quantity which tends to 0 as x approaches 0. We won't compute $o()$. $o()$, this is the error term when we approximate $f(x)$ by $P_n(x)$.

Proposition 2.
This polynomial P_n in the Taylor expansion $DL_n(0)$ of f is UNIQUE. and denoted by $[f]_n$.

Example 2.
Find $DL_2(0)$ of $f(x) = 1 + 3x - 5x^2 + 12x^3 + 5x^4$
Video : example 2

Proposition 3.
If f is even (respectively odd) then $[f]_n$ is even (respectively odd).

2.2 Taylor expansion and differentiable functions

Theorem 4 (Mac-Laurin)**.**
Let's assume that $n \geqslant 1$. If $f \in \mathcal{C}^{n-1}(I)$, such that $f^{(n)}(0)$ exists, f has a $DL_n(0)$ given by its Mac-Laurin serie

$$f(x)=\sum_{k=0}^{n}\left[\frac{f^{(k)}(0)}{k!}x^k\right]+o\left(x^n\right)$$
$$=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+o\left(x^n\right)$$

Remark 2.

1. f has an expansion $DL_0(0)$, iif f is continuous at 0. Then
$$\forall x\in I, f(x)=f(0)+o(1)$$

2. f has an expansion $DL_1(0)$ iif f is differentiable at 0. Then
$$\forall x\in I, f(x)=f(0)+xf'(0)+o(x)$$

3. there exists functions that do not satisfy Taylor Young's theorem but that get an expansion for $n\geqslant 2$

Example 3.

Let f be the function defined by $f(x)=\begin{cases} x^3\sin\dfrac{1}{x}\ si\ x\neq 0\\ 0\ si\ x=0\end{cases}$
Prove that f has an expansion $DL_2(0)$, but the second order derivative of f does not exist at 0.
Video : example 3

2.3 Common Taylor serie

$e^x=$
$\cos x=$

$\sin x=$

$\operatorname{ch} x=$

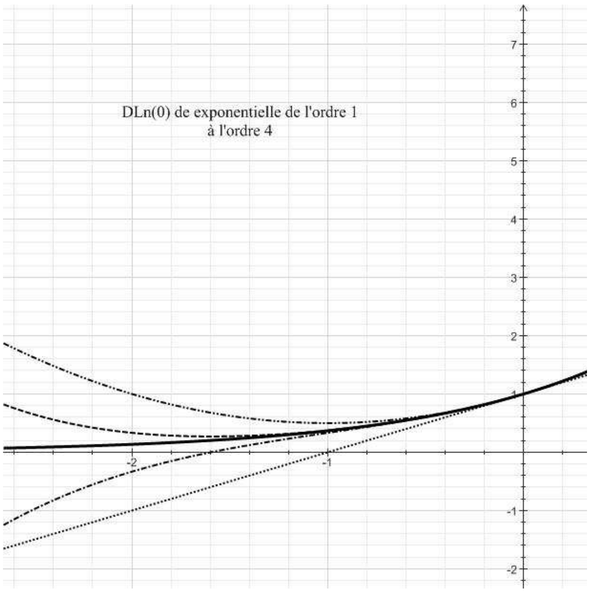
$\operatorname{sh} x=$

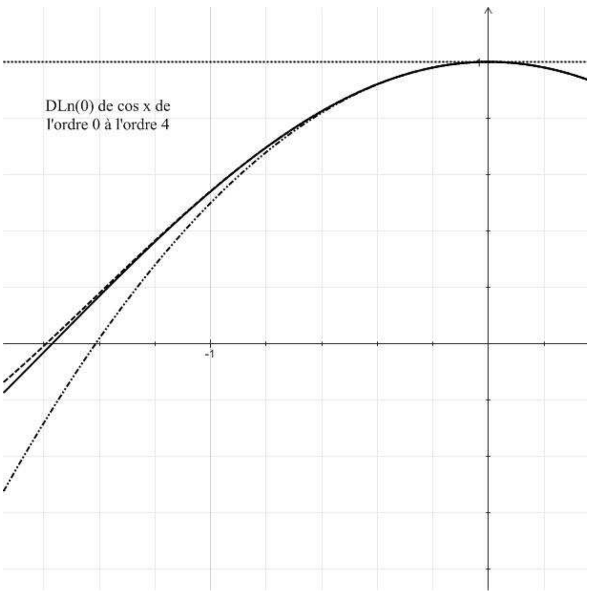
$(1+x)^\alpha=$

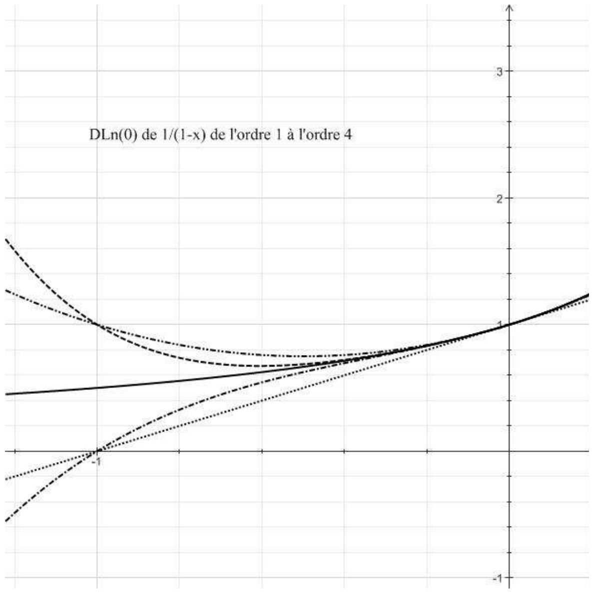
$\dfrac{1}{1+x}=$

$\dfrac{1}{1-x}=$

Video : for the exponential function
Video : for the sine and the cosine
Video : for the hyperbolic sine and the hyperbolic cosine
Video : other functions







2.4 Operations on taylor expansions

1. **Linear combination**
   Let f and g be two functions of $\mathcal{F}(I,\mathbb{R})$ and $\lambda \in \mathbb{R}$. We assume that both f and g get Taylor series $DL_n(0)$ then $f+\lambda g$ has a Taylor expansion $DL_n(0)$ and : $[f+\lambda g]_n = [f]_n + \lambda[g]_n$.
2. **Multiplication**
   Let f and g be two functions $\mathcal{F}(I,\mathbb{R})$. We assume that both f and g get series expansions $DL_n(0)$ then $f.g$ has a series expansion $DL_n(0)$ and we get : $[f.g]_n = [[f]_n.[g]_n]_n$.
   Example 4.
   (a) Give $DL_3(0)$ of $\dfrac{e^x}{1+x}$
   (b) Give $DL_n(0)$ of $\dfrac{1}{(1-x)^2}$
   Video : example 4 a)
   Video : example 4 b)
3. **Composition**
   Let f be a function of $\mathcal{F}(I,\mathbb{R})$ and g a function $\mathcal{F}(J,\mathbb{R})$ such that $f(I) \subset J$. We assume that $\boxed{f(0)=0}$ then $g \circ f$ has a series expansion $DL_n(0)$ and we get : $[g\circ f]_n = [g]_n \circ [f]_n$.
   Example 5.
   Give $DL_4(0)$ of $f(x) = e^{\cos x}$
   Video : example 5
4. **The inverse**
   Let g be a function of $\mathcal{F}(I,\mathbb{R})$ getting a series expansion $DL_n(0)$ such that $\boxed{g(0) \neq 0}$

then $\dfrac{1}{g}$ has a series expansion $DL_n(0)$ get by increasing power order. The division by increasing power order is useful to compute series expansion.

Example 6.
Give $DL_5(0)$ of $f(x) = \dfrac{1}{\operatorname{ch} x}$
Video : example 6

5. **The division**
   Let f and g be two functions of $\mathcal{F}(I, \mathbb{R})$. We assume that f and g get series expansions $DL_n(0)$ such that $\boxed{g(0) \neq 0}$ then $\dfrac{f}{g}$ has a series expansion $DL_n(0)$ get using a division by increasing power.

   Example 7.
   Give $DL_5(0)$ of $f(x) = \tan x$
   Video : example 7

2.5 Integration

Let $f \in \mathcal{C}^0(I)$ having a series expansion $DL_n(0)$ given by $f(x) = \sum_{k=0}^{n} \left[\dfrac{f^{(k)}(0)}{k!}x^k\right] + o(x^n)$. Then all antiderivative F of f has a serie expansion on I $DL_{n+1}(0)$ given by :

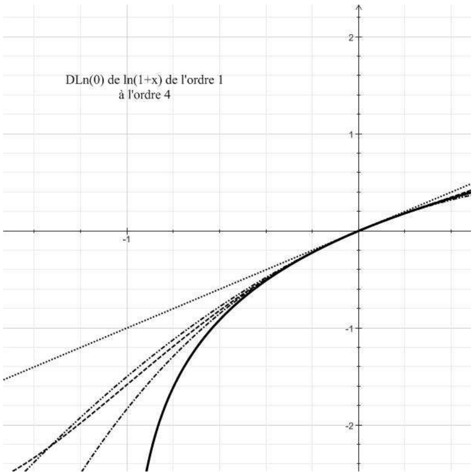
$$F(x) = F(0) + \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\frac{x^{k+1}}{k+1} + o(x^{n+1})$$

The simplest method is to integrate "term by term" the series expansion $DL_n(0)$ of f and to add the constant $F(0)$ Thus we have

$$\ln(1+x) = \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k} + o(x^n) = x - \frac{x^2}{2} + \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$$

$$\operatorname{Arctan} x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})$$

Video : example



Example 8. On your own
Give the $DL_3(0)$ of $\text{Arcsin}\, x$

2.6 Derivative

Let $f \in \mathcal{C}^0(I)$ be a function getting a series expansion $DL_n(0) : f(x) = \sum_{k=0}^{n} \left[\frac{f^{(k)}(0)}{k!} x^k \right] + o\left(x^n\right)$.
Then the series expansion $DL_{n-1}(0)$ of f', **if it exists** is given by :

$$f'(x) = \sum_{k=1}^{n} \frac{1}{(k-1)!} f^{(k)}(0) x^{k-1} + o(x^{n-1})$$

Example 9. On your own
Prove that the derivative f' of the function : $f(x) = \begin{cases} x^2 \sin \frac{1}{x} \ si \ x \neq 0 \\ 0 \ si \ x = 0 \end{cases}$ has no expansion at 0 $DL_0(0)$ whereas f has an expansion $DL_1(0)$

2.7 Series expansions at the neighborhood of a

Definition 3.
f has a serie expansion truncated at order n at the neighborhood of a, denoted by $DL_n(a)$ if and only if there exist a real polynomial P_n of degree less or equal than n, such that : $f(x) - P_n(x-a) = o((x-a)^n)$ at the neighborhood of a.
The expansion $DL_n(a)$ of f is written :

$$\begin{aligned} f(x) &= P_n(x-a) + o((x-a)^n) \\ &= a_0 + a_1(x-a) + \cdots + a_n(x-a)^n + o((x-a)^n) \end{aligned}$$

Remark 3.
In practice using a change of variables we will compute a series expansion $DL_n(0)$ at 0. To find the expansion $DL_n(a)$ of $f(x)$ we set $h = x - a \Leftrightarrow x = a + h$ and thus will compute an expansion in h at 0.

Example 10.
Find the serie expansion truncated at 3 of $f(x) = sin(x)$ at $\frac{\pi}{2}$.

Careful : a serie expansion $DL_n(a)$ is a polynomial expression in $x - a$. We won't developp powers of $x - a$.

Taylor Mac-Laurin's formula is a particular case of the following formula true for any real number a :

Theorem 5 (Taylor-Young formula)**.**
Let's assume that $n \geqslant 1$. Let $f \in \mathcal{C}^{n-1}(I)$, such that its n-th order derivative $f^{(n)}(a)$ exists. Then f has a series expansion $DL_n(a)$ given by Taylor-Young formula :

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + o((x-a)^n)$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o((x-a)^n)$$

3 Asymptotic expansions

Definition 4.
f has an expansion truncated at order n in the neighborhood of $+\infty$ (respectively $-\infty$), called an asymptotic expansion of f and denoted by $DL_n(+\infty)$ (respectively $DL_n(-\infty)$ if and only if there exists a polynomial E and a polynomial P_n of degree less or equal tahn n, such that :
$f(x) - \left(E(x) + P_n\left(\frac{1}{x}\right)\right) = o\left(\frac{1}{x^n}\right)$ in the neighborhood of $+\infty$ (respectively $-\infty$).
An asymptotic expansion, that is to say a $DL_n(\pm\infty)$ of f is written :
$f(x) = E(x) + P_n\left(\frac{1}{x}\right) + o\left(\frac{1}{x^n}\right) = E(x) + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n} + o\left(\frac{1}{x^n}\right)$.

We set a change of variables $x = \frac{1}{t}$ which means $t = \frac{1}{x}$. Then we look for an expansion $DL_n(0)$, then we go back to $f(x)$.

Example 11.
Find the asymptotic expansion at $+\infty$ truncated at order 2 of $f(x) = \sqrt{x^2 + 5x + 1}$

4 Equivalence

Definition 5.
Let f and f be two functions $\mathcal{F}(I, \mathbb{R})$. f is asymptotically equivalent to g in the neighborhood of a where $a \in [-\infty, +\infty]$, if and only if : $f - g = o(g)$ in the neighborhood of a. We denote $f \underset{a}{\sim} g$ ou $f(x) \underset{x \to a}{\sim} g(x)$.

Proposition 6 (Fundamental Characterisation)**.**
If $g \neq 0$ in the neighborhood of a then we get :

$$f \underset{a}{\sim} g \Leftrightarrow \frac{f(x)}{g(x)} \underset{x \to a}{\to} 1$$

Example 12.
Find the following equivalents at 0 :

1. $e^x \sim 1 + x$
2. $e^x \sim 1 + 2x$
3. $e^x - 1 \sim 2x$

   Video : example

Remark 4.
From the preceding example, we note that the equivalent of a function is not unique. On the other hand, the equivalents can not be easily manipulated.

Proposition 7. If $f \sim g$ and $l \sim k$ in the neighborhood of a then $\lim_{x \to a} \frac{f}{l} = \lim_{x \to a} \frac{g}{k}$

4.1 Fundamental examples at 0

If f has a series expansion $DL_n(0)$ then $f \underset{0}{\sim} [f]_n$:

$e^x - 1 \underset{0}{\sim}$ $\ln(1+x) \underset{0}{\sim}$ $(1+x)^\alpha \underset{0}{\sim}$

$\cos x \underset{0}{\sim}$ $\sin x \underset{0}{\sim}$ $\tan x \underset{0}{\sim}$

Video : example

On your own

$\operatorname{sh} x \underset{0}{\sim}$ $\operatorname{th} x \underset{0}{\sim}$ $\operatorname{Arcsin} x \underset{0}{\sim}$

$\operatorname{Arctan} x \underset{0}{\sim}$ $\operatorname{Argsh} x \underset{0}{\sim}$ $\operatorname{Argth} x \underset{0}{\sim}$

1. All non zero polynomial is asymptotically equivalent at $+\infty$ or $-\infty$ to its higher degree term.
2. All non zero polynomial is asymptotically equivalent at 0 to its lower degree term.
3. All non zero rational fraction is asymptotically equivalent at $+\infty$ or $-\infty$, to the quotient of its higher degree terms.
4. All non zero rational fraction is asymptotically equivalent a 0, to the quotient of its lower degree terms.

It is possible to MULTIPLY equivalents, however it is forbidden to add them.

5 Applications

5.1 Applications of series and asymptotic expansions

Here is a list of the main applications :

1. *To compute a limit.*
2. *To find the equation of a tangent at a point*
   If f has a serie expansion $DL_n(a)$ of the shape $f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + o((x-a)^n)$ then $y = a_0 + a_1(x-a)$ is the equation of the tangent line at $(a, f(a))$ and its position is given by the sign of the first non zero element following $a_1(x-a)$.

Example 13.
Compute $\lim_{x\to 0} \frac{x(1+\cos x) - 2\tan x}{2x - \sin x - \tan x}$
Video : example
Video : example
On your own, prepare exercise 6) 1),2),3).

Example 14.
Give the equation of the tangent line at 1 for the function $Arctan x$ and give its position.
Video : example

5.2 Asymptotic expansions and asymptotic behaviour

Asymptotic equation.
Using a change of variables, by setting $x = \frac{1}{t}$ meaning $t = \frac{1}{x}$. We are looking for an expansion $DL_n(0)$ and then go back to $f(x)$. If f has an expansion $DL_n(\pm\infty)$ of the shape : $f(x) = a_0 x + a_1 + \frac{a_p}{x^p} + o\left(\frac{1}{x^p}\right)$ then $y = a_0 x + a_1$ is an oblique asymptote for f en $\pm\infty$. The sign of $\frac{a_p}{x^p}$ gives the position of the graph to the asymptote.

Example 15.
Find the equation of the oblique asymptote at the graph $y = \sqrt{\frac{x^3}{x-1}}$ and give the position of the graph to the asymptote.

Video : example

Exercises

Exercise 1.
Give the $DL_3(0)$ of those fuunctions :

1. $f(x) = \sin x + \cos x$
2. $b(x) = \sin x \ln(1+x)$
3. $g(x) = e^{2x}$
4. $h(x) = x \ln(x+1) - x$
5. $a(x) = \dfrac{x^2+1}{x^2+2x+2}$
6. $i(x) = \dfrac{\sin x}{x}$
7. $j(x) = \dfrac{\sqrt{x+1}-1}{x}$
8. $k(x) = \ln(1+\sin x)$
9. $l(x) = \dfrac{\arcsin x}{\sqrt{1-x^2}}$
10. $c(x) = \ln\left(\dfrac{1}{\cos x}\right)$
11. $m(x) = \dfrac{1}{x} - \dfrac{1}{\sin x}$
12. $n(x) = \sqrt[3]{1+x}$ and $o(x) = \sqrt[3]{1-x^2}$

Exercise 2.
Let $f(x) = \begin{cases} (1+x^2) + x^2\varepsilon(x) \text{ si } x \neq 0 \\ 1 \text{ si } x = 0 \end{cases}$,
where $\varepsilon(x) = x \sin \dfrac{1}{x}$.
Show that f amits a $DL_2(0)$ that does not comes from Mac-Laurin formula.

Exercise 3.
Give a $DL_3(1)$ of $f(x) = \sqrt{x}$

Exercise 4.
Give a $DL_3(+\infty)$ of

1. $f(x) = \sqrt[3]{x^3+1} - (x+1)$
2. $g(x) = \dfrac{x^3+2}{x-1}$

Exercise 5.

1. Give a $DL_2(+\infty)$ of $\dfrac{x+1}{x+2}$
2. Give a $DL_2(+\infty)$ of $\sqrt{\dfrac{x+1}{x+2}}$
3. Give a $DL_2(0)$ of $\operatorname{Arctan} x$
4. Give a $DL_2(+\infty)$ of $\operatorname{Arctan}\left(\sqrt{\dfrac{x+1}{x+2}} - 1\right)$

Exercise 6.
Find those limits at 0 using a $DL_n(0)$:

1. $f(x) = \dfrac{\sin x - x\cos x}{x(1-\cos x)}$
2. $f(x) = \dfrac{\sin x - \tan x}{x^3}$
3. $f(x) = \dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}$

4. $f(x) = \dfrac{1}{x} - \dfrac{1}{\ln(1+x)}$
5. $f(x) = \dfrac{1}{x}\left(\dfrac{1}{\operatorname{th} x} - \dfrac{1}{\tan x}\right)$
6. $f(x) = \dfrac{1}{x}\ln\left(\dfrac{e^x - 1}{x}\right)$
7. $f(x) = \dfrac{\cos x}{\ln(1+x)}$

Exercise 7.
Find those limits at 1 using a $DL_n(1)$:

1. $f(x) = \dfrac{1}{\ln x} - \dfrac{x}{\ln x}$
2. $f(x) = \dfrac{1 - x + \ln x}{1 - \sqrt{2x - x^2}}$
3. $f(x) = \dfrac{e^x - e^{1/x}}{x^2 - 1}$

Exercise 8.

1. Give the asymptotic expansion of order 3 for $\ln x - \ln(x-1)$.
2. Deduce this limit : $\lim\limits_{x \to +\infty} \dfrac{1}{e^x}\left(\dfrac{x}{x-1}\right)^{x^2}$

Exercise 9.
Calculate this limit : $\lim\limits_{x \to +\infty} \left(1 + \dfrac{1}{x}\right)^x$

Exercise 10.
Give equations of the tangent lines at 0, as well as the relative position of the curve and its tangent line in the neighborhood of 0

1. $f(x) = \dfrac{\sin(x)}{x}$
2. $g(x) = \dfrac{e^x - 1 - x}{x\sin(x)}$

Exercise 11.
Using Taylor expansions at infinity, determine the equation of the asymptotes to those graphs :

1. $y = \sqrt{x^2 + 4x - 5}$
2. $y = x^2 \ln\left(\dfrac{x-1}{x}\right)$
3. $y = e^{-\frac{1}{x}}\sqrt{x^2 + 1}$

Exercise 12.
Give equivalents for :

1. $\ln x$ at 1.
2. $\ln^4(1+x)$ at 0.
3. $\dfrac{\sin x}{x}$ at 0.
4. $\dfrac{x^2 + 3}{x^4 + 2}$ at $+\infty$.
5. $\dfrac{e^{-x} + 2}{x^2 + x^4}$ at $+\infty$.