$\boxed{\text{LINEAR MAPS}}$

Objectifs
— **Define linear maps.**
— **understand image and kernel of a linear map.**
— **work on linear maps in finite dimension.**

In this chapter we use $\mathbb{K}$ wich represents either $\mathbb{R}$ or $\mathbb{C}$.

1 Generalities

Definition 1.
Let E and E' be two K vector spaces. Let f be a map from E to E'. f is a linear map (or a module homomorphism) if and only if it checks those properties :
(i) $\forall x, y \in E, f(x+y) = f(x) + f(y)$
(ii) $\forall x \in E, \forall \lambda \in K, f(\lambda \cdot x) = \lambda \cdot f(x)$
This means that f matches the structure of K vector space of E to E'.

Example 1.
Are the following maps linear ?

1. Let E be $\mathbb{K}$ vector space and $k \in \mathbb{K}$. The mapping from E into $E : x \mapsto k \cdot x$ is called homothety of factor k.
2. The mapping from $\mathbb{R}$ into $\mathbb{R}$ such that $x \mapsto x^2$.

Property 1.
If f is a linear mapping from E into E' then $f(0_E) = 0_{E'}$.

Example 2.

1. Prove that property.
2. Is the converse true ?

Remark 1.
To show that a mapping is not linear, we can use the contraposition of the previous property, namely, if we have $f(0_E) \neq 0_{E'}$ then f is not linear.

Theorem 1 (Practical Theorem)**.**
Let f be a map from E to E', two K vector spaces.
f is a linear map if and only if $\forall x, y \in E, \forall \alpha \in K$:

$$f(\alpha x + y) = \alpha f(x) + f(y)$$

Example 3.

1. Is the mapping from $\mathbb{R}^2$ into $\mathbb{R}^3$, defined by $(x, y) \mapsto (x - y, 0, y)$ a linear mapping ?
2. Prove the previous theorem.

Definition 2.
Let E be a vector space of K. A linear form on E is a linear map from the vector space E to its field of scalars K.

Example 4.
Are those maps linear forms ?

1. The map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which maps (x, y) to $2(x, y)$.
2. The map from $\mathbb{R}^2$ to $\mathbb{R}$ which maps (x, y) to $x^2 + y^2$.
3. $f \mapsto \displaystyle\int_0^1 f(t)dt$ where $f \in \mathcal{C}^0([0, 1])$

2 Operations on linear maps

Definition 3.
We denote $\mathcal{L}(E, E')$ the set of linear maps of the vector-space E over K in the dans vector space E' over K.

Theorem 2.
Let f, g be two linear maps from E into E' and $k \in \mathbb{K}$. Then $f + g$ and kf are linear maps from E into E'.

Proposition 3. $\mathcal{L}(E, E')$ is a vector space over K, as a sub-space of the vectoriel space of maps between E to E'.

Proposition 4. The composition of two linear maps is a linear map.

Example 5.
Prove the following theorem.

3 Endomorphisms

Definition 4.
Let E be a vector space over K. An endomorphism of E is a linear map from E to itself. We denote by $\mathcal{L}(E)$ the set of endomorphisms of E

Remark 2.
For endomorphisms, we use this noattion : $f \circ f \circ f = f^3$.

Example 6.
Why f^2 has no meaning if f is the linear map from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(x, y) = x$?

4 Isomorphisms and automorphisms

Definition 5.
Let f be a linear map from E to E' two vector spaces over $\mathbb{K}$.

1. f is an isomorphism if and only if f is bijective.
2. f is an automorphism if and only if f is an endomorphism and is bijective, so is both an endomorphism and an isomorphism.

Theorem 5.
The inverse of an isomorphism is an isomorphism.

Example 7.
— Is the vectoriel homothety of E of factor k an automorphism ? If yes, give its inverse.
— Is this map $(x, y) \mapsto x + iy$ an isomorphism between $\mathbb{R}^2$ and $\mathbb{C}$? An automorphism ?
— Prove the previous theorem.

5 Kernel and image (or range)

5.1 Kernel

Example 8.
Let f be a linear map.
We already know that $f(0_E) = 0_{E'}$.

1. Is it possible to find other vectors u such that $f(u) = 0_{E'}$?
2. Prove that f is injective if and only if 0_E is the only vector u of E satisfying $f(u) = 0_{E'}$.

Definition 6.
Let E and E' be two vector spaces over K and let f be a linear map from E to E'. The **kernel** of f is the set :

$$Kerf = f^{-1}(\{0_{E'}\}) = \{x \in E / f(x) = 0_{E'}\}$$

Example 9.

1. Let's consider $u : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (y, x + y + z)$. Find the kernel of u.
2. Let's consider $u : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto (2x - y, x + 2y, x + y)$. Find the kernel of u.

Theorem 6.
The kernel of a linear map from E to E' is a vector sub-space of E.

Example 10.
Prove the previoud theorem.

From the previous example, we deduce the following theorem :

Theorem 7.
Let f be a linear map from E to E' then f is injective if and only if : $Kerf = \{0_E\}$

5.2 Image

Definition 7.
Let E and E' be two vector spaces over K and f a linear map from E to E'. The **image** (or range) is the set :

$$Imf = f(E) = \{f(x)/x \in E\}$$

Example 11.
Find the image of the following linear maps :

1. Soit $u : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (y, x + y + z)$.
2. Soit $u : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto (2x - y, x + 2y, x + y)$.

Theorem 8.
The image of a linear map from E to E' is a vcetor sub-space of E'.

Example 12.
Prove the previous theorem.

Theorem 9.
Let E and E' be two vector spaces over K and $f : E \to E'$ a linear map.

If $S = (e_1, \ldots, e_p)$ is a spanning set of E, which means $E = Vect(e_1, \ldots, e_p)$ then $S' = (f(e_1), \ldots, f(e_p))$ is a spanning set of Imf.

Remark 3.
This theorem allows to find the image of f Imf using only a spanning set of E.

Example 13.

1. With the previous theorem, find the image of the following linear maps :
   (a) Soit $u : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (y, x + y + z)$.
   (b) Soit $u : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto (2x - y, x + 2y, x + y)$.
2. Prove the previous theorem.

6 Linear maps in finite dimension

6.1 Linear maps and family of vectors

Theorem 10.
Let E and E' be two vectr spaces ove K and $f : E \to E'$ a linear map.

1. f is injective $\Leftrightarrow$ the image under f of any linearly independent family of vectors of E is a linearly independent of E' : let $B = (e_1, \ldots, e_p)$ be a linearly independent family of vectors of E, f is injective $\Leftrightarrow$ $(f(e_1), \ldots, f(e_p))$ is also a linearly independent family of vectors of E'.
2. f is surjective $\Leftrightarrow$ the image under f of all spanning set of E is a spanning set of E' which means : let $B = (e_1, \ldots, e_p)$be any spanning set of E, f is surjective $\Leftrightarrow$ $(f(e_1), \ldots, f(e_p))$ is a spanning set of E'.
3. f is bijective $\Leftrightarrow$ the image under f of any basis of E is a basis of E' which means : let $B = (e_1, \ldots, e_p)$ be a basis of E, f is bijective $\Leftrightarrow$ $(f(e_1), \ldots, f(e_p))$ is also a basis of E'.

6.2 Rank nullity theorem

Theorem 11.
Let f be a linear map from E to E', then :

$$\dim \operatorname{Ker} f + \dim \Im m f = \dim E$$

Remark 4.

1. Let's denote that $dim Imf \leqslant dimE$
2. Due to the rank nullity theorem the dimension of the codomain has no influence

Example 14.
Write the rank nullity theorem for this map $u : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (0, x + y)$.

Example 15.
Prove the rank nullity theorem.

6.3 Rank for a linear map

Definition 8.
Let E and E' be two finite dimensional $\mathbb{K}$ vector spaces and f a linear map from E to E'. We call rank of f the dimension of $\Im m f$.

Remark 5.
Thus, the theorem of rank is also written : $\operatorname{rg}(f) = \dim E - \dim \operatorname{Ker} f$

Theorem 12.
Let $(e_1, \ldots, e_n)$ be a basis of E. Then for all linear map f from E to E' we have : $\operatorname{rg}(f) = \operatorname{rg}(f(e_1), \ldots, f(e_n))$

Example 16.
Let f the function defined on$\mathbb{R}^3$ by $f(x, y, z) = (x + y, y + z, 2x + y - z)$

Determine the rank of this functions using two methods $(f(\vec{i}), f(\vec{j}), f(\vec{k}))$ where $(\vec{i}, \vec{j}, \vec{k})$ is a basis of $\mathbb{R}^3$.

Theorem 13.
Let E and E' be two $\mathbb{K}$ vector spaces of finite dimensiosn and f A linear mapping of E into E' then we have the following equivalences :

— f is injective $\Leftrightarrow \operatorname{rg}(f) = \dim E$
— f is surjective $\Leftrightarrow \operatorname{rg}(f) = \dim E'$
— f is bijective $\Leftrightarrow \dim E = \operatorname{rg}(f) = \dim E'$

6.4 How to characterize isomorphisms

Theorem 14.
Let E and E' be two finite dimensional vector spaces over K with **the same** dimension and f a linear map from E to E'. The following sentences are equivalent :

i) f is injective.

ii) f is surjective.

iii) f is bijective.

And therefore its corollary :

Corollary 15.
Let E be a vector space over K of finite dimension, f an endomorphism of E dans E.
We get : f is an automorphism of E $E \Leftrightarrow Kerf = \{0_E\} \Leftrightarrow Imf = E$

Example 17.
Prove that the map f from $\mathbb{R}^2$ to itself defined by : $f(1,0) = (2,2)$ et $f(0,1) = (1,3)$ is an automorphism of $\mathbb{R}^2$.

Example 18.
Let

$$f : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^3 \\ (x,y) \mapsto (x, x+y, y) \end{cases}$$

Show that f is injective but not surjective.

7 Exercises

Exercise 1.
Which of the following mappings are linear ?

$$f_1 : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^2 \\ (x,y,z) \mapsto (x-z, x+y) \end{cases}$$

$$f_2 : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^3 \\ (x,y,z) \mapsto (xz, x, x+z) \end{cases}$$

$$f_3 : \begin{cases} C^1(\mathbb{R}) \to C^0(\mathbb{R}) \\ f \mapsto f + f' \end{cases}$$

$$f_4 : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^3 \\ (x,y) \mapsto (x+y, x, y) \end{cases}$$

$$f_5 : \begin{cases} C^0(\mathbb{R}) \to C^0(\mathbb{R}) \\ f \mapsto \displaystyle\int_a^x f(t)dt \end{cases}$$

$$f_6 : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ (x,y) \mapsto (x+1, y) \end{cases}$$

$$f_7 : \begin{cases} \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R}) \\ f \mapsto 2f \end{cases}$$

Exercise 2.
Are the following linear forms ?

1. The null mapping of E in $\mathbb{K}$.
2. $(x,y) \mapsto ax + by$ where $(x,y,a,b) \in \mathbb{R}^4$.
3. Let u_0 be a vector of $\mathbb{R}^2$. The mapping which for all u of $\mathbb{R}^2$ associates it's scalar product with u_0.

Exercise 3.

For linear maps in exercise 1, determine their kernel and image. Specify whether the functions are injective and / or surjective.

Exercise 4.
Let p be the map defined by : $p : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ (x,y) \mapsto (4x-6y, 2x-3y) \end{cases}$

1. Show that p is linear
2. Show that p is a projection ie $p \circ p = p$.
3. Determine Ker p et Im p.
4. Is p injective, surjective ?

Exercise 5.
Let $\mathbb{R}^2$ have it's canonical basis $\left(\vec{i}, \vec{j}\right)$ and $\mathbb{R}^4$ have it's canonical basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$. Let $\phi : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by :

$$\phi(x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3 + t\vec{e}_4) = (x + y + 2z + t)\,\vec{i} + (2x - y + 2z - 7t)\,\vec{j}$$

Assuming ϕ is a linear mapping, determine Ker ϕ and Im ϕ.

Exercise 6.
Let f be a linear mapping from $\mathbb{R}^2$ into $\mathbb{R}^5$, defined by $x = (\alpha, \beta)$ of $\mathbb{R}^2$:

$$f(x) = (\alpha + 2\beta, -2\alpha + 3\beta, \alpha + \beta, 3\alpha + 5\beta, -\alpha + 2\beta)$$

. We admit that f is a linear map.

1. Determine $\text{Ker}\,(f)$ and its dimension.
2. Determine $\Im m\,(f)$ and its dimension.

Exercise 7.
Considering the vector space $E = C^{\infty}(\mathbb{R})$, let $f_1(x) = e^x,\ f_2(x) = e^{2x},\ \ f_3(x) = e^{3x}$.

1. Determine the dimension of the vector subspace F of E defined by $F = Vect\,(f_1, f_2, f_3)$
2. Let $\phi : F \to F$, be defined by $\forall f \in F, \phi(f) = f'' + f' - 3f$. show that ϕ is an endomorphism of F.
3. Is ϕ an automorphism ?

Exercise 8.
Let f be a function from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $f : (x, y) \mapsto (x + y, x - y)$.

1. Show that f is an automorphism of $\mathbb{R}^2$.
2. determine its inverse.

Exercise 9.
Let E and E' be two finite-dimensional vector spaces, and f be a linear mapping of E into E'.
Are the following statements true or false ?

1. It is possible to have non-bijective f and $\dim E = \dim E'$.
2. It is possible to have non-bijective f and $\dim E = \dim \text{Im}\, f$.
3. It is possible to have f non bijective and $\dim E' = \dim \text{Im}\, f$.
4. If $\text{rg}\, f = 5$ and $\dim E' = 3$, then we don't know $\dim \text{Ker}\, f$.
5. If $\dim E$=5, and f surjective then $\dim E'$=5.
6. If $\mathcal{F} = (u_1, u_2, u_3)$ is a linearly dependant set of E, then $f(\mathcal{F})$ is a linearly dependant set of E'.
7. If $\mathcal{F} = (u_1, u_2)$ is a linearly independant set of E, then $f(\mathcal{F})$ is a linearly independant set of E'.

Exercise 10.
Let a, b, c real numbers with $c \neq 0$. We consider in $\mathbb{R}^3$, the vector : $w = (\,a,\ b,\ c\,)$.
Let $\mathcal{B}_c = (\,\overrightarrow{i}\,,\,\overrightarrow{j}\,,\,\overrightarrow{k}\,)$ be a basis of $\mathbb{R}^3$.
Let f be an endomorphism of $\mathbb{R}^3$ such that for all vectors $t = (\,x,\ y,\ z\,)$ of $\mathbb{R}^3$ $f\ \ (t) = (\,c\,y - b\,z,\ a\,z - c\,x,\ b\,x - a\,y\,)$.

1. Show that $w \in \text{Ker}\ (\,f\,)$.
2. Show that the set $\left(f\,(\,\overrightarrow{i}\,),\ f\,(\,\overrightarrow{j}\,)\right)$ is linearly independant.
3. Deduce that $\text{Ker}\,(\,f\,) = Vect\ (\,w\,)$ and determine a basis of $\Im m\,(\,f\,)$.
4. Is f injective ? Futhermore $(\vec{i}, \vec{j})$ and $\left(f\,(\,\overrightarrow{i}\,),\ f\,(\,\overrightarrow{j}\,)\right)$ are not collinear. Is this in contradiction with 1) of theorem 8 ?

Exercise 11.
Let $(\vec{i}, \vec{j}, \vec{k})$ a basis of $\mathbb{R}^3$ and f a mapping of $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by :
$f(x, y, z) = (y - x, y + z, x)$.

1. Show that $f$is an automorphism of $\mathbb{R}^3$.
2. Give the rank of f.
3. Let $F = \mathcal{V}ect(f(\vec{\imath}), f(\vec{\jmath}))$ and $G = \mathcal{V}ect(f(\vec{\imath}), f(\vec{k}))$.
   Without any calculation determine $F \cap G$.

Exercise 12.

In $\mathbb{R}^2$, we define an endomorphism u by :
$\forall (x, y) \in \mathbb{R}^2, \quad u(x, y) = (2x - y, x + y)$.

1. What is the rank of u ? Deduce that u is an automorphism.
2. Let $X = (x, y)$ be a vector of $\mathbb{R}^2$.
   (a) Determine the image of X by $u \circ u$.
   (b) What can be said of the set $(X, u(X), u \circ u(X))$? Deduce three non zero reals $\alpha, \beta, \varepsilon$ independent of x and y such that : $\alpha u \circ u(X) + \beta u(X) + \varepsilon X = 0$.
   (c) Deduce that the endomorphism $v = \alpha u \circ u + \beta u + \varepsilon Id$ is the null endomorphism.
   (d) Composing v by u^{-1}, deduce u^{-1} as function of u and Id. Determine the coordinates of $u^{-1}(X)$ as a function of x and y.

Exercise 13. (optional)
Let f and g be two endomorphisms of $\mathbb{K}$ vector space E.
Show that $\Im m\,(g \circ f) \subset \Im m\,(g)$ and $\operatorname{Ker}(f) \subset \operatorname{Ker}(g \circ f)$.

Exercise 14. (optional)
Let E be a Kk vector space of dimension 3. Let g be an endomorphism of E satisfying $g^2 \neq 0$ and $g^3 = 0$.

1. Check the following inclusions : $0_E \subset \operatorname{Ker} g \subset \operatorname{Ker} g^2 \subset E$.
2. Show that $1 \leqslant \dim \operatorname{Ker} g \leqslant 2$

Exercise 15. (optional)
Let F and G be two vector subspaces of a vector space E of finite dimension.

1. Considering $\phi : \begin{cases} F \times G \to E \\ (x, y) \mapsto x + y \end{cases}$ et $\psi : \begin{cases} F \cap G \to F \times G \\ x \mapsto (x, -x) \end{cases}$.
   (a) Show that ϕ and ψ are linear mappings
   (b) On what conditions on F and G, is ϕ an isomorphism ?
   (c) Compare $\operatorname{Ker} \phi$ and $\operatorname{Im} \psi$.
   (d) Justify $\dim \operatorname{Im} \psi = \dim F \cap G$.
2. Show that $\dim F \times G = \dim F + \dim G$.
3. Deduce, using the rank formula, a proof of the Grassmann formula :

$$\dim F + G = \dim F + \dim G - \dim F \cap G$$