Inverse functions

0.1 Inverses of trigonometric functions

0.1.1 The Arcsine function

Video : The Arcsine function

Definition 1.
The restriction of the sine function from $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ to [-1 ;1] is continuous and strictly increasing, thus it has an inverse function defined on $[-1;1]$ and takes values on $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ called the Arcsine function and denoted by Arcsin. This function is strictly increasing on $[-1;1]$, continuous on $[-1;1]$ but differentiable on $]-1;1[$ as the derivative of the sine function is null at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Remark 1 (BE CAREFUL).

The function Arcsin is not the inverse function of the sine function but of its restriction to $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$. We get for instance :

$$\sin\left(\operatorname{Arcsin}\frac{1}{3}\right) = \frac{1}{3}$$
$$\operatorname{Arcsin}\left(\sin\frac{\pi}{8}\right) = \frac{\pi}{8}$$

but
$\operatorname{Arcsin}\left(\sin\frac{3\pi}{4}\right) = \frac{\pi}{4}$ as it is the unique element of $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ whose sine is equal to the sine of $\frac{3\pi}{4}$.

Example 1. On your own
Compute $\operatorname{Arcsin}(\sin(\pi/3))$, $\operatorname{Arcsin}(\sin(4\pi/3))$

Derivative
Video : Derivative

$$\forall x \in]-1;1[, \operatorname{Arcsin}' x = \frac{1}{\sin'(\operatorname{Arcsin} x)} = \frac{1}{\cos(\operatorname{Arcsin} x)}$$

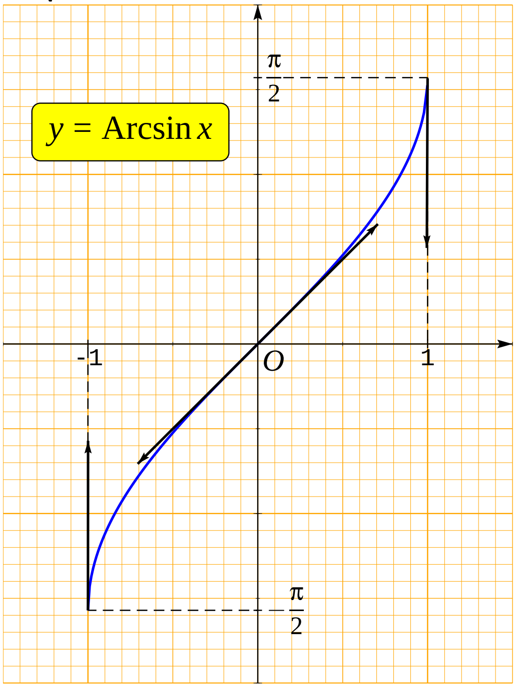
but $\cos^2(\operatorname{Arcsin} x) + \sin^2(\operatorname{Arcsin} x) = 1$
thus it follows $\cos^2(\operatorname{Arcsin} x) = 1 - x^2$.
As $\operatorname{Arcsin} x \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ we get $\cos(\operatorname{Arcsin} x) \geqslant 0$ so $\cos(\operatorname{Arcsin} x) = \sqrt{1-x^2}$. Finally :

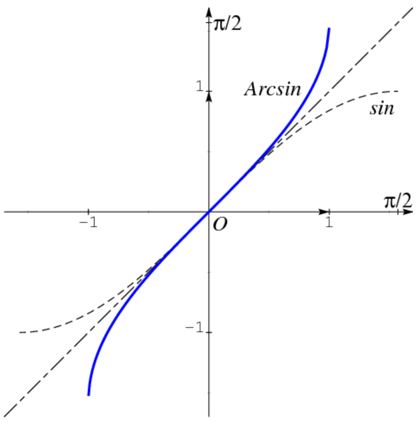
$$\forall x \in]-1;1[, \operatorname{Arcsin}' x = \frac{1}{\sqrt{1-x^2}}$$

Oddness Evenness

Arcsin is odd as it is the inverse function of an odd function. So it sufficies to study the function on $[0;1]$.

Graph





0.1.2 The Arccosine function

Video : The Arcosine function

Definition 2. The restriction of the cosine function to $[0;\pi]$ takes values in $[-1;1]$ and is both continuous and strictly decreasing, thus it has an inverse function defined on $[-1;1]$ which takes values in $[0;\pi]$. This function is called the Arccosine and denoted by Arccos. This function is strictly decreasing on $[-1;1]$, continuous on $[-1;1]$ but differentiable on $]-1;1[$ as the derivative of the cosine function is null at 0 and π.

Remark 2 (BE CAREFUL).

Arccos is not the inverse function of the cosine function but of its restriction to $[0;\pi]$. Thus for instance we get :

$$\cos\left(\text{Arccos}\,\frac{2}{3}\right) = \frac{2}{3}$$
$$\text{Arccos}\left(\cos\frac{\pi}{5}\right) = \frac{\pi}{5}$$

but

$\text{Arccos}\left(\cos\frac{4\pi}{3}\right) = \frac{2\pi}{3}$ as it is the unique element of $[0;\pi]$ such that its cosine is the same as the cosine of $\frac{4\pi}{3}$.

Video : Fundamental remark

Example 2. On your own
Compute Arcsin(cos(−π/2)), Arcsin(cos(π/2))

Derivative
Video : Derivative

$$\forall x \in]-1;1[, \operatorname{Arccos}' x = \frac{-1}{\sqrt{1-x^2}}$$

Example 3.

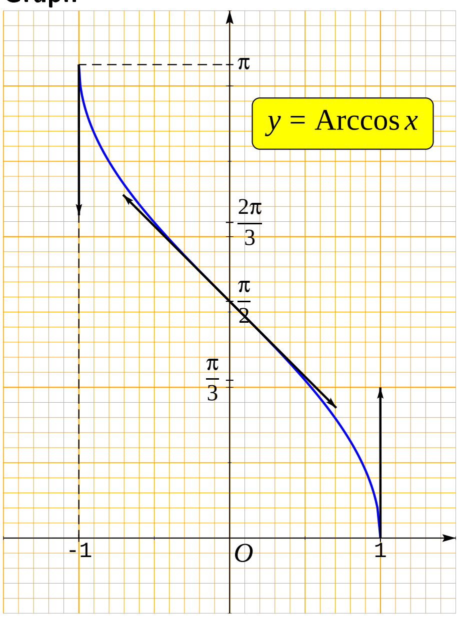
Prove it

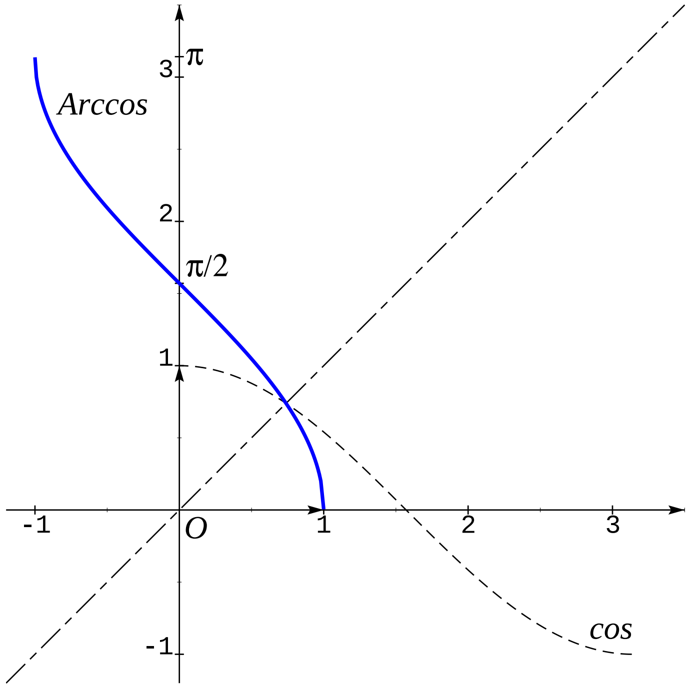
$$\forall x \in]-1;1[, \operatorname{Arccos}' x = \frac{1}{\cos'(\operatorname{Arccos} x)} = \frac{-1}{\sin(\operatorname{Arccos} x)}$$

but $\cos^2(\operatorname{Arccos} x) + \sin^2(\operatorname{Arccos} x) = 1$
so $\sin^2(\operatorname{Arccos} x) = 1 - x^2$.
Since $\operatorname{Arccos} x \in [0;\pi]$ we have $\sin(\operatorname{Arccos} x) \geqslant 0$ we get $\sin(\operatorname{Arccos} x) = \sqrt{1-x^2}$. So finally :

$$\forall x \in]-1;1[, \operatorname{Arccos}' x = \frac{-1}{\sqrt{1-x^2}}$$

Video : Derivative
Be careful
Arccos is neither odd nor even.
Graph





Example 4.

Prove using two different ways that : $\forall x \in [-1;1], \operatorname{Arccos} x + \operatorname{Arcsin} x = \dfrac{\pi}{2}$

Video : Example

On your own :

1. Prove that $\cos(\operatorname{Arccos} x + \operatorname{Arcsin} x) = 0$.
2. Prove that $\sin(\operatorname{Arccos} x + \operatorname{Arcsin} x) = 1$.
3. Justify that $\forall x \in [-1, 1]$, $(\operatorname{Arccos} x + \operatorname{Arcsin} x) \in [-\pi/2, 3\pi/2]$.
4. Deduce that $\forall x \in [-1;1], \operatorname{Arccos} x + \operatorname{Arcsin} x = \dfrac{\pi}{2}$.

0.1.3 Arc tangent

Video : The Arctan function

Definition 3. The restriction of the tangent function to $\left]-\dfrac{\pi}{2};\dfrac{\pi}{2}\right[$ takes values in $]-\infty;+\infty[$ and is continuous and stricly increasing, thus it has an inverse function defined on $]-\infty;+\infty[$, which takes values in $\left]-\dfrac{\pi}{2};\dfrac{\pi}{2}\right[$. This inverse function is called the arctangent function and denoted by Arctan. This inverse function is strictly increasing on $\mathbb{R}$, continuous and differentiable on $\mathbb{R}$.

Remark 3 (BE CAREFUL).

Arctan is not the inverse function of the tangent function but the inverse function of its restriction to $\left]-\dfrac{\pi}{2};\dfrac{\pi}{2}\right[$. That is why we get for instance :

$$\tan(\text{Arctan}\,5) = 5$$
$$\text{Arctan}\left(\tan\frac{\pi}{7}\right) = \frac{\pi}{7}$$

but
$\text{Arctan}\left(\tan\frac{8\pi}{7}\right) = \frac{\pi}{7}$ as it is the unique element of $\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ such that its tangent is the same as the tangent of $\frac{8\pi}{7}$.

Example 5. On your own
Compute $\text{Arctan}(\tan(-\pi/3))$, $\text{Arctan}(\tan(2\pi/3))$

Derivative

$$\forall x \in \mathbb{R}, \text{Arctan}'\,x = \frac{1}{1+x^2}$$

Example 6.

Prove the formula for the derivative

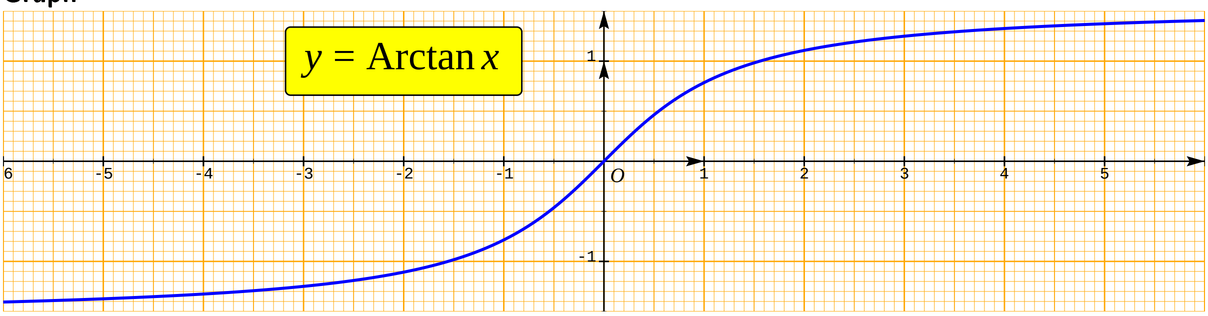
$$\forall x \in \mathbb{R}, \text{Arctan}'\,x = \frac{1}{\tan'(\text{Arctan}\,x)} = \frac{1}{1+\tan^2(\text{Arctan}\,x)}$$

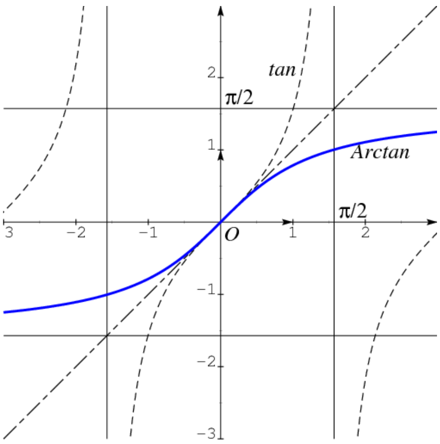
Thus it follows :

$$\forall x \in \mathbb{R}, \text{Arctan}'\,x = \frac{1}{1+x^2}$$

Video : Derivative

Oddness Evenness
Arctan is an odd function as it is the inverse function of an odd function.
Graph





Example 7. Prove that :

1. $\forall x > 0, \operatorname{Arctan} x + \operatorname{Arctan} \frac{1}{x} = \frac{\pi}{2}$
2. $\forall x < 0, \operatorname{Arctan} x + \operatorname{Arctan} \frac{1}{x} = -\frac{\pi}{2}$

Video : Example

0.2 Inverse functions of hyperbolic functions

Definition 4.
The hyperbolic cosine ch is a bijection (one to one correspondence) from $[0;+\infty[$ to$[1;+\infty[$ as it is continous and sctictly increasing, thus its inverse function exists and is called inverse hyperbolic cosine :

$$\operatorname{Argch} x : [1,+\infty[\to [0,+\infty[$$

Definition 5.
The hyperbolic sine sh is a bijection from $\mathbb{R}$ to $\mathbb{R}$ as it is continuous and strictly increasing from $\mathbb{R}$ to $\mathbb{R}$, thus its inverse function exists and is called inverse hyperbolic sine.

$$\operatorname{Argsh} x : \mathbb{R} \to \mathbb{R}$$

Definition 6.
The hyperbolic tangent th is continuous and stricly increasing from $\mathbb{R}$ to $]-1;1[$, thus ist inverse function exists and is called inverse hyperbolic tangent.

$$\operatorname{Argth} x :]-1,1[\to \mathbb{R}$$

Video : Introduction

Example 8.

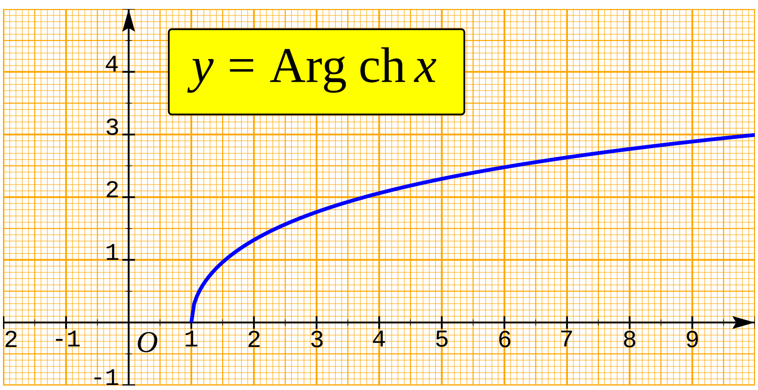
1. Let's define $f(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ for all $x \geqslant 1$
   (a) Calculate $\mathrm{ch}(f(x))$ for $x \geqslant 1$.
   (b) Calculate $f(\mathrm{ch}(x))$ for $x \geqslant 0$.
   (c) What is your conclusion ?
   Video : Example
2. Let's define for all $x \in \mathbb{R}$, $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$
   (a) We set $y = f(x)$. Express x depending on y.
   (b) What is your conclusion ?
   Video : Example
3. We admit that $x \in]-1; 1[$, $\mathrm{Argth}\, x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$

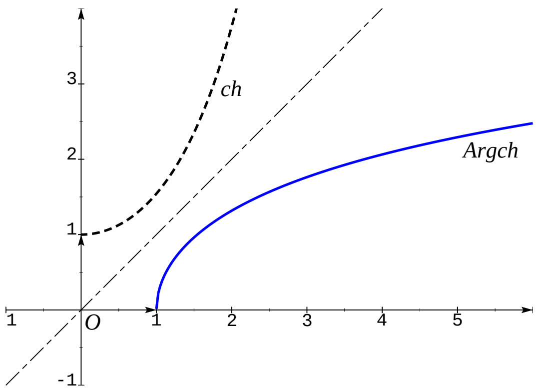
Example 9. Prove that :

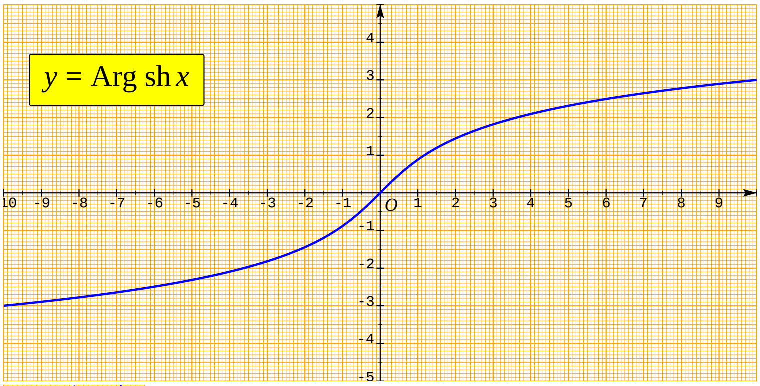
1. For all $x > 1$, $\mathrm{Argch}'\, x = \frac{1}{\sqrt{x^2 - 1}}$
   Video : Derivative 1
2. For all $x \in \mathbb{R}$, $\mathrm{Argsh}'\, x = \frac{1}{\sqrt{x^2 + 1}}$
   Video : Derivative 2
3. For all $x \in]-1; 1[$, $\mathrm{Argth}'\, x = \frac{1}{1 - x^2}$
   Video : Derivative 3

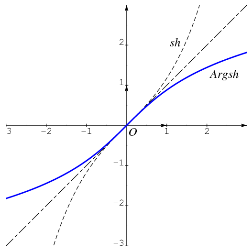
Example 10.
Find $\mathrm{ch}(\mathrm{Argch}\, x)$ for $x \in [1; +\infty[$ and $\mathrm{Argch}(\mathrm{ch}\, x)$ for $x \in \mathbb{R}$, distinguishing cases if necessary.
Video : Example

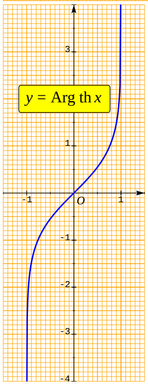
Example 11.
Study the map $f : x \mapsto x\, \mathrm{Argsh}\, x$.
Video : Example

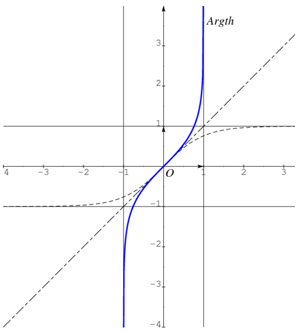












Exercises

Exercise 1.

1. Compute when possible : $\sin(\arcsin(2))$, $\arccos(\cos(2\pi))$, $\arcsin(\sin(\frac{10\pi}{3}))$.
2. Simplify, specifying the domain of definition : $\cos(\arcsin(x))$, $\cos(\arctan(x))$ and $\tan(\arcsin x))$.

Exercise 2.
Solve this equation :
$\arcsin(\sqrt{3}x) = \frac{\pi}{2} - \arcsin x$

Exercise 3.
Study those functions :

$$f(x) = \sin(\arcsin x)\, \text{et}\, g(x) = \arcsin(\sin x)$$

Exercise 4. Let's define : $f(x) = \arcsin\left(\frac{2x}{1+x^2}\right) - 2\arctan x$

1. Give the domain of definition of f ?
2. Compute f'. Let's deduce an easier expression for f.
3. Draw the graph of f.

Exercise 5.

1. Compute if possible : $\text{Argch}(1)$, $\text{Argch}(\text{ch}(-2))$, $\text{sh}(\text{Argch}(2))$, $\text{Argsh}(\text{sh}(-10))$, $\tan(\arctan(\frac{\pi}{2}))$.

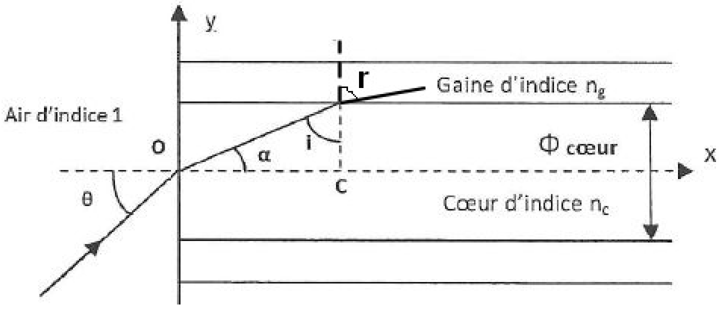
2. Simplify (specifying the domain of definition) : $\text{ch}(\text{Argch}(x))$

Exercise 6. 1. Show that the equation $\text{Argch}(x) + \text{Argsh}(x) = 1$ eventually has a solution by expliciting it.

2. Why do you have the word "eventually" in the previous question ?
3. Show, by studying a function, that the equation has a solution.
4. Conclude.

Exercise 7.
Study this function : $\arctan(\text{th}(\frac{x}{2}))$

Exercise 8. (optional)
The diagram below shows an optical fiber, with $n_c > n_g$.



According to Descartes' laws for refraction : $\sin\theta = n_c \sin\alpha$ and $n_c \sin I = n_g \sin r$.
If $\frac{n_c}{n_g}\sin I > 1$, the ray is reflected on the sheath, and $r = \frac{\pi}{2}$, is then the limiting case of refraction.
Show that $\sin\theta = \sqrt{n_c^2 - n_g^2}$ if $r = \frac{\pi}{2}$.
$\sin\theta$ is called the numerical aperture.