MAPS

Learning objectives and outcomes

- **To know few fundamental proofs.**
- **To know the definition of a map.**
- **To be able to prove if a map is one to one, onto or bijective.**
- **To be able to find the direct image and the inverse image of a set under a map.**

1 MAPS

1.0.1 Definitions

Definition 1.
Let E and F be two sets, a map f from E to F is a rule (relation) that assigns a **unique** element y denoted by $f(x)$ in F to each element x in E.

$$f \begin{cases} E \to F \\ x \mapsto y = f(x) \end{cases}$$

E is called thed domain of f and F its codomain.
Let x be an element of E ; the element $y = f(x)$ is called **the image** of x under f.
Let $y \in F$, if there exists an element x such that $y = f(x)$ then x is called **the preimage or the fiber** of y by f. Pay attention to the fact that there could exist several fibers or preimages for y.
Let's denote $\mathcal{F}(E; F)$, the set of maps from E to F.

The words "functions" and "maps" are synonymous. In analysis we generally prefer the first one rather than the other one. We will soon speak about linear maps which is why we use in this chapter the word map.

Thus, to define a map f requires three points :

- a domain E
- a codomain F
- a process to match x to $f(x)$

If one of those parameters is changed then we get a new map, even though the others are the same. Usually the domain E is not known. Using the expression of $f(x)$, we will have to find real numbers x such that $f(x)$ is well-defined. This set is called the **domain of definition**.

Proposition 1.
In order to prove that two maps f and g are equal, we have to check that they have :
- the same domain E
- the same codomain F
- the same values, which means : $\forall x \in E, f(x) = g(x)$

Example 1.
Let $f : x \mapsto x^2$ be a map from $\mathbb{R}_+$ to $\mathbb{R}$ and $g : x \mapsto x^2$ a map from $\mathbb{R}$ to $\mathbb{R}$ then $f \neq g$ as the domains of definition are different.

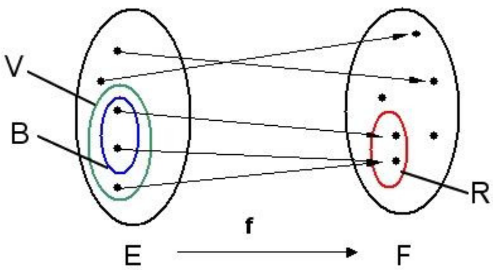
1.1 Direct image and inverse image

Definition 2.
Let f be a map from E to F. If A is a subset of E, the direct image of A under f, is the subset of F containing the images under f of all elements of A.
We denote by$f(A)$ this set.
An element y of F is in $f(A)$ if and only if there exists x in A such that $y = f(x)$.
Mathematically :

$$f(A) = \{y \in F / \exists x \in A, y = f(x)\} = \{f(x) / x \in A\}$$

In the particular case if $A = E$, the set $f(E)$, range of E by f is called the range of f and denoted by $\operatorname{Im} f$.

Example 2.
Find $f(B)$ and draw $f(E)$.



Definition 3.
Let f be a map from the set E to F. If B is a subset of F,the inverse image of B under the map f, is the subset of E containing elements x of E such that $f(x) \in B$.
We denote $f^{-1}(B)$ this set. Thus, an element x of E is in $f^{-1}(B)$ if and only if its image $f(x)$ is in B. Mathematically speaking :

$$f^{-1}(B) = \{x \in E / f(x) \in B\}$$

Example 3.
Find $f^{-1}(R)$ and draw $f^{-1}(F)$ in the previous example.

Remark 1. Method.
Let $f \in \mathcal{F}(E, F)$,

1. To find the inverse image of a set B of the codomain F by f, it sufficies to solve $f(x) \in B$
2. To find the direct image of a subset A of the domain E by f (or to find the image set $f(A)$) , it sufficies to find elements $y \in F$ such that the equation $(y = f(x) \, and \, x \in A)$ has at least one solution.

Example 4.
Let's consider the map f from $\mathbb{R}$ to $\mathbb{R}$, $f \colon x \mapsto x^2$.

1. Find $f([-3, -1])$,
2. Find $f^{-1}(]-\infty, 2])$,
3. Let E and F be two sets and f a map from E to F. Let a and B two subsets of E prove that $f(A \cap B) \subset f(A) \cap f(B)$ but that the equality is not true.

1.2 Function composition

Definition 4.
Let E, F and G be three sets, f be a map from E to F and g a map from F to G. We denote $g \circ f$ the map from E to G to yield a map which maps x from E, to $(g \circ f)(x) = g(f(x))$. The map $g \circ f$ is g composed with f.

Remark 2.
Usually $g \circ f \neq f \circ g$!

Property 1.

1. Let's consider $f \in \mathcal{F}(G, H), g \in \mathcal{F}(F, G), h \in \mathcal{F}(E, F)$ then : $(f \circ g) \circ h = f \circ (g \circ h)$
2. Let's consider $f \in \mathcal{F}(E, E), f \circ Id_E = Id_E \circ f = f$

Example 5.
Give $g \circ f$ and $f \circ g$ with $E = F = G = \mathbb{R}, f(x) = x^2 - 1, g(x) = 2x - 3$

1.3 Surjection

Definition 5.
Let f be a map from E to F. f is surjective or onto if it checks one the following equivalent property :

1. For every element y of F, the equation $y = f(x)$ has at least one solution,
2. $\forall y \in F, \exists x \in E$ such that $y = f(x)$
3. For every element of the codomain F is mapped to by at least one element of the domain by f.

Example 6.
Let's define the map f from $\mathbb{R}$ to $\mathbb{R}$ by $f(x) = x^3 - 3x$. Is this map surjective ?

Property 2.
Let f be a function from E to F. Then f is a surjection from E to $f(E)$.

Example 7.
Prove the previous property.

1.4 Injection

Definition 6.
Let f be a map from the set E to the set F. f is an injective function or injection or one to one function if it checks one of the following equivalent property :

1. For every element y of F, the equation $y = f(x)$ has at most one solution.
2. $\forall (x_1, x_2) \in E^2, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
3. $\forall (x_1, x_2) \in E^2, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
4. Every element of the function's codomain F is the image of at most one element of its domain by f.

A one-to-one function is a function that preserves distinctness : it never maps distinct elements of its domain to the same element of its codomain. The term one-to-one function must not be confused with one-to-one correspondence.

Property 3.
Let I be a part of $\mathbb{R}$ and let f be a strictly monotonous function on I. Then f is an injection from I to $\mathbb{R}$.

Example 8.

1. The map f from $\mathbb{R}$ to $\mathbb{R}$ is defined by $f(x) = x^3 - 3x$. Is f injective ?
2. Is the converse of the previous property true ?

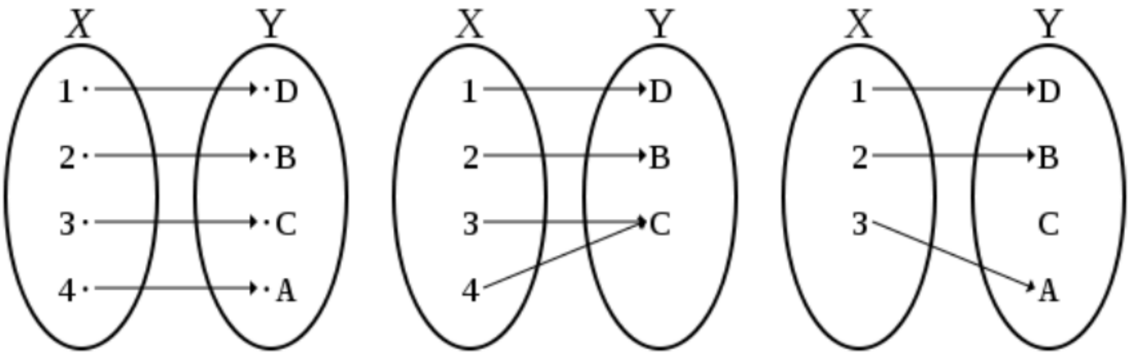
1.5 Bijection

Definition 7.
Let f be a map from E to F. f is said bijective or one to one and onto or one to one correspondence if it checks one of the following equivalent property :

1. For every element y of the codomain F, the equation $y = f(x)$ has a unique solution,
2. $\forall y \in F, \exists! x \in E$ such that $y = f(x)$
3. Every element of the codomain F is the image by f of exactly one element of the domain
4. f is both one to one and onto/

Remark 3.
Usually a map is neither one to one (injective) nor onto (surjective).

Example 9.
Find below where is the injection, the surjection and the bijection :



Property 4.
Let f be a real-valued, continuous function and strictly monotonic on an interval I. Then f is a bijection.

1.6 Examples

1. The map f from $\mathbb{R}$ to $\mathbb{R}$ which assigns to every real x the value $f(x) = e^x$ is injective or one to one (as it is a continuous and strictly increasing function) but it is not onto (a negative real number has no fiber under f)
2. The map f from $\mathbb{R}$ to $\mathbb{R}$ which assigns to every real x the value $f(x) = \sin(x)$ is neither one to one as $\forall x \in \mathbb{R}, \sin(x) = \sin(x + 2\pi)$ nor onto asany real number whose absolute value is strictly greater than 1 has no fiber under f.

1.7 Inverse function

Definition 8.
Let f be a bijection from E to F. The inverse function of f, denoted by f^{-1} is the map from F vers E, defined for all y de F by,
$x = f^{-1}(y) \Rightarrow y = f(x)$

Theorem 2.
If $f \in \mathcal{F}(E, F), g \in \mathcal{F}(F, E)$ are such that $f \circ g = Id_F$ and $g \circ f = Id_E$ then those functions are both bijective and inverse functions of each other.

Method In order to prove that a map f belonging to $\mathcal{F}(E, F)$ is bijective and to find its inverse function we may :

1. Either find a map $g \in \mathcal{F}(F, E)$ such that $f \circ g = Id_F$ and $g \circ f = Id_E$

2. Or solve the equation $y = f(x)$ and prove that whatever is $y \in F$, this equation has a unique solution $x = f^{-1}(y)$

Example 10.
Let f be the function defined on $\mathbb{R}$ by $f(x) = 2x + 5$.

1. Justify without any computation that f is a bijection from $\mathbb{R}$ to $\mathbb{R}$.
2. Find the inverse function g of f.
3. Prove that g is the inverse function of f.

1.8 Functions from $\mathbb{R}$ to $\mathbb{R}$

Property 5. Odd functions If f is a bijection from I to J such that both I and J are symmetric with respect to 0 then if f is odd so is f^{-1}.

Example 11. Prove the previous property

Property 6. Monotonicity of the inverse function
Let f be a bijective function from a set I to a set J strictly increasing (respectively strictly decreasing), then its inverse function is a strictly increasing function on J(respectively strictly decreasing).

Property 7. *Derivative of the inverse function*
Let f be a bijective and differentiable function from I to J.

$$\forall y \in J \ / \ f'(f^{-1}(y) \neq 0, (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Example 12.
Using the fact that the exponential function is the inverse of the logarithmic function, find the derivative of the exponential function.

Exercises

Exercise 1.

1. Find $f(A)$ in the following cases : $f \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$ et $A = \left[-\frac{1}{2}, 2\right]$.
2. Let's define f by $f(x) = \dfrac{x-2}{x-1}$. Give its domain of definition D. Find $f(D)$.

Exercise 2.
Compute, if it is possible, $f^{-1}(0)$ and $f^{-1}(\{0\})$ for the following functions defined on $\mathbb{R}$:

1. $f(x) = x - 1$
2. $f(x) = x^2 - 1$

Exercise 3.
Let E and F be two parts of $\mathbb{R}$. Let's consider the map f from E to F defined by :
$f(t) = \frac{1}{2}t^2 + 3t + 2$.

1. Here $E = \mathbb{R}$ and $F = \mathbb{R}$. Draw the graph (C) of f in the cartesian plane $\left(O, \vec{i}, \vec{j}\right)$.
2. Using those graphs, find $f(\mathbb{R})$, $f(\mathbb{R}^+)$, $f(]-4;1])$, $f^{-1}(\mathbb{R})$, $f^{-1}(\mathbb{R}^+)$ et $f^{-1}([-3;2])$.
3. In the following cases, is the map f one to one, onto or bijective ?
   (a) $E = \mathbb{R}$ and $F = \mathbb{R}$
   (b) $E = [-3, +\infty[$ and $F = \mathbb{R}$;
   (c) $E = [-3, +\infty[$ and $F = \left[-\frac{5}{2}; +\infty\right[$
   .

Exercise 4.
Let $f\begin{cases}\mathbb{N} \to \mathbb{N} \\ x \mapsto x+1\end{cases}$ and $g\begin{cases}\mathbb{N} \to \mathbb{N} \\ y \mapsto 0 \text{ if } y = 0\text{ , } y-1 \text{ ifnot}\end{cases}$

1. Study whether f or g is one to one, onto or bijective, if possible give f^{-1} and g^{-1}.
2. Give an expression for $g \circ f$ and $f \circ g$.

Exercise 5.
Are those maps one to one, onto or bijective. If it is a bijection give its inverse function :

1. $f : \begin{array}{c}\mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{x}{1+|x|}\end{array}$
2. $f : \begin{cases}\mathbb{R}^2 \to \mathbb{R}^2 \\ (x,y) \mapsto (2x-y, x+y)\end{cases}$
3. $f : \begin{cases}\mathbb{R}^2 \to \mathbb{R} \\ (x,y) \mapsto 2x+y-1\end{cases}$
4. $f : \begin{cases}\mathbb{R} \to \mathbb{R}^2 \\ x \mapsto (x+1, 2x+1)\end{cases}$

Exercise 6.
Prove that the relationship $y = x + \sqrt{x^2+1}$ defines a map f de $\mathbb{R} \to \mathbb{R}_+^*$
Prove that f is bijective and find its inverse function.

Exercise 7.
Is this map $f : \mathbb{C} \setminus \{0\} \to \mathbb{C},\ z \mapsto z + \frac{1}{z}$ one to one ? onto ? bijective ?
Find theimage by f of the circle of center 0 and radius 1.
Find the inverse image by f of the straight line $i\mathbb{R}$.

Exercise 8.
Let's define $f \in \mathcal{F}(E, F)$ and $g \in \mathcal{F}(F, G)$ two maps. Prove that :

1. If f and g are one to one then $g \circ f$ is one to one
2. If f and g are onto then $g \circ f$ is onto
3. If f and g are bijective then $g \circ f$ is bijective.

Exercise 9. (Optional)
Let's consider those four sets A, B, C and D and the maps $f : A \to B$, $g : B \to C$, $h : C \to D$. Prove that :

$$g \circ f \text{ one to one} \implies f \text{ one to one,}$$
$$g \circ f \text{ onto} \implies g \text{ onto.}$$

Montrer que :

$$\Big(g \circ f \text{ and } h \circ g \text{ are bijective }\Big) \Leftrightarrow \Big(f, g \text{ and } h \text{ are bijective}\Big).$$

Exercise 10. (Optional)
Let's define $f : X \to Y$. We denote $\hat{f} : \begin{cases} \mathcal{P}(X) \to \mathcal{P}(Y) \\ A \mapsto f(A) \end{cases}$ et $\tilde{f} : \begin{cases} \mathcal{P}(Y) \to \mathcal{P}(X) \\ B \mapsto f^{-1}(B) \end{cases}$. Prove that

1. f is one to one iif $\hat{f}$ is one to one.
2. f is onto iif $\tilde{f}$ is one to one.