MATRIX AND MATRIX ALGEBRA : applications to two by two matrices

Learning objectives

- **To understand what a matrix is.**
- **To be able to add and multiply matrices**
- **To compute a determinant and the inverse of a two by two matrix.**

Throughout the chapter we will designate by $\mathbb{K}$ the sets $\mathbb{R}$ or $\mathbb{C}$.

1 Matrices

1.1 Definitions

Definition 1 (Notation)**.**
We call matrix of n rows and p columns, an array of np numbers belonging to $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. So a matrix is a rectangular collection of numbers.
We denote it by :

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & \cdots & a_{np} \end{pmatrix} = (a_{ij})_{\substack{1\leqslant i\leqslant n \\ 1\leqslant j\leqslant p}}$$

under extended or compressed form.
The a_{ij} are real or complex, i is the index for the row and j for the column of a_{ij}.
$\mathcal{M}_{n,p}(\mathbb{K})$ denotes the set of matrices with n rows and p columns, with coefficient in the set $\mathbb{K}$.

When $p = 1$ we have a column vector : $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

Example 1.
Each of the following matrices belong to $\mathcal{M}_{n,p}(\mathbb{K})$. Determine n, p and $\mathbb{K}$ for each one :

1. $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$
2. $B = \begin{pmatrix} i & -i \\ -i & \sqrt{2} \\ 1-i & 2 \end{pmatrix}$
3. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
4. $D = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$

Video : Generalities and example 1

1.2 Basic operations

1.2.1 Matrix addition

The addition of two matrices is a very natural process. To perform matrix addition, two matrices must have **the same dimensions** . In that case simply add each individual components, like below. It is simply denoted by + and we have the following definition :

Definition 2.
Let A and B two matrices with **to the same dimensions** $\mathcal{M}_{n,p}(\mathbb{K})$, then we get :

$$A = (a_{ij})_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant p}}$$

$$B = (b_{ij})_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant p}}$$

then

$$A + B = (a_{ij} + b_{ij})_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant p}} \in \mathcal{M}_{n,p}(\mathbb{K})$$

Remark 1.
The addition of two matrices is only possible if the two matrices belong to the same set. Otherwise, the sum does not exist !

Example 2.
Soit $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}$ et $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \\ 1 & 2 \end{pmatrix}$.
Justify that the sum of the two matrices is possible and compute $A + B$.

Video : addition and example 2

Property 1.
Let A, B et C be three matrices with the same dimensions $\mathcal{M}_{n,p}(\mathbb{K})$

(i)

$$A + B = B + A$$

Matrix addition is commutative.

(ii)

$$(A + B) + C = A + (B + C)$$

Matrix addition is associative.

(iii) The identity element is called the zero matrix (or null matrix) denoted by O such that :

$$O = (0)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant p}} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

(iv) Each matrix $A \in \mathcal{M}_{n,p}(\mathbb{K})$ owns a symmetry matrix denoted by $-A \in \mathcal{M}_{n,p}(\mathbb{K})$ such that :

$$A + (-A) = O$$

We denote $A - A = O$. Thus we get :

$$-A = (-a_{ij})_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant p}}$$

With those properties, $(\mathcal{M}_{n,p}(\mathbb{K}), +)$ is a commutative group.

1.2.2 Matrix multiplication and multiplication by a scalar

It is possible to multiply a matrix by a scalar belonging to $\mathbb{K}$. To multiply a matrix by a scalar, also known as scalar multiplication, multiply every element in the matrix by the scalar.

Definition 3.

Let's consider $A \in \mathcal{M}_{n,p}(\mathbb{K})$ such that $A = (a_{ij})_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant p}}$ and a scalar $\alpha \in \mathbb{K}$; then we get :

$$\alpha \cdot A = \alpha A = (\alpha a_{ij})_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant p}} = \begin{pmatrix} \alpha a_{11} & \cdots & \alpha a_{1p} \\ \vdots & \alpha a_{ij} & \vdots \\ \alpha a_{n1} & \cdots & \alpha a_{np} \end{pmatrix} \in \mathcal{M}_{n,p}(\mathbb{K})$$

Example 3.
Let's consider $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R})$ and let $\alpha = 7$.
Compute $7A$.

Video : multiplication by a scalar and example 3

Remark 2.
The scalar is always written to the left of the matrix. Thus it is written $7A$ but certainly not $A7$! It is also written $\frac{1}{7}A$ but certainly not $\frac{A}{7}$!

Property 2. Some properties on the law $\cdot$

(i)
$$\forall \alpha \in \mathbb{K}, \forall (A,B) \in (\mathcal{M}_{n,p}(\mathbb{K}))^2, \alpha(A+B) = \alpha A + \alpha B$$

(ii)
$$\forall (\alpha,\beta) \in \mathbb{K}^2, \forall A \in \mathcal{M}_{n,p}(\mathbb{K}), (\alpha+\beta)A = \alpha A + \beta A$$

(iii)
$$\forall (\alpha,\beta) \in \mathbb{K}^2, \forall A \in \mathcal{M}_{n,p}(\mathbb{K}), \alpha(\beta A) = (\alpha\beta)A$$

(iv) 1 is the identity element for the scalar multiplication whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$

Remark 3.
Due to the four previous properties, the set $(\mathcal{M}_{n,p}(\mathbb{K}), +, \cdot)$ is a vector space. The vectors are in this vector space matrices with n rows and p columns. Once again we see that the scoring with an arrow would be disastrous!

1.2.3 Matrix multiplication

Matrix multiplication is not a very natural law.

Definition 4.
Let a matrix $A = (a_{ij})_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant \mathbf{p}}} \in \mathcal{M}_{n,\mathbf{p}}(\mathbb{K})$ and $B = (b_{ij})_{\substack{1\leqslant i\leqslant \mathbf{p}\\1\leqslant j\leqslant q}} \in \mathcal{M}_{\mathbf{p},q}(\mathbb{K})$.

It is possible to multiply A by B, we have

$$A \times B \in \mathcal{M}_{n,q}(\mathbb{K})$$

and :

$$A \times B = (c_{ij})_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant q}}$$

with

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Example 4. $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$ Compute AB and BA.

Video : multiplication and example 3

Remark 4. VERY IMPORTANT !

1. The number of columns of the first matrix in the multiplication should be equal to the number of rows of the second matrix. Otherwise, the calculation of $A \times B$ is impossible.
2. The result of the matrix product of two matrices, has the number of rows of the first matrix and the number of columns of the second.
3. We can write AB instead of $A \times B$

Property 3.

1. There exit matrices $A \in \mathcal{M}_{n,n}$ and $B \in \mathcal{M}_{n,n}$ such that $AB \neq BA$.
2. There exist matrices $A \in \mathcal{M}_{n,n}$ and $B \in \mathcal{M}_{n,n}$, A and B nonzero, such that $AB = 0$

Said otherwise, generally $AB \neq BA$ (We say that the matrix product is not commutative) and $AB = 0 \nRightarrow A = 0$ or $B = 0$.

Property 4. Some other properties

(i)

$$\forall A \in \mathcal{M}_{n,p}(\mathbb{K}), \forall B \in \mathcal{M}_{p,q}(\mathbb{K}), \forall C \in \mathcal{M}_{q,r}(\mathbb{K}), A(BC) = (AB)C$$

We say that the matrix product is associative.

(ii)

$$\forall A \in \mathcal{M}_{n,p}(\mathbb{K}), \forall (B_1, B_2) \in (\mathcal{M}_{p,q}(\mathbb{K}))^2, A(B_1 + B_2) = AB_1 + AB_2$$

We say that the matrix product is distributive with respect to the matrix sum.

(iii)

$$\forall A \in \mathcal{M}_{n,p}(\mathbb{K}), \forall B \in \mathcal{M}_{p,q}(\mathbb{K}), \forall \alpha \in \mathbb{K}, (\alpha A) B = \alpha (AB)$$

2 Two by two matrices

2.1 Links between matrices and systems

$\begin{cases} ax+by &= c \\ a^{'}x+b^{'}y &= c^{'} \end{cases} \Leftrightarrow \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ c' \end{bmatrix} \Leftrightarrow MX = Y$

with $M = \begin{bmatrix} a & b \\ a' & b' \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $Y = \begin{bmatrix} c \\ c' \end{bmatrix}$

Example 5. Determine the system or find the matrix attached :

1. $\begin{cases} 2x+4y &= 5 \\ x-y &= 3 \end{cases}$
2. $\begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

Video : Example 5

2.2 Inverse of a two by two matrix

Definition 5. Let M be a two by two matrix, M is invertible means there exists a matrix N such that

$$MN = NM = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the inverse matrix of M is denoted M^{-1} and the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ called identity matrix is denoted I_2.

Example 6. Show that M and N are inverse :

$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$N = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

Definition 6. Let $M = \begin{bmatrix} a & b \\ a' & b' \end{bmatrix}$, the determinant of M, denoted by $\det M$, is equal to

$$\det M = ab' - ba'$$

Property 5. M is invertible if and only if $\det(M) \neq 0$

Example 7. Is The matrix $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ invertible ?

Property 6. Let $M = \begin{bmatrix} a & b \\ a' & b' \end{bmatrix}$ an invertible matrix, then

$$M^{-1} = \frac{1}{\det M} \begin{bmatrix} b' & -b \\ -a' & a \end{bmatrix}$$

Example 8. Compute the inverse matrix of M.

Video : Examples 6, 7, 8

2.3 To solve a system

Consider the system
$$\begin{cases} ax+by &= c \\ a'x+b'y &= c' \end{cases}$$
Let M be the matrix attached to this system, if M is invertible then we get a unique solution (x,y) :
$$\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1} \begin{bmatrix} c \\ c' \end{bmatrix}$$

Example 9. Solve the following system : $\begin{cases} 2x+y &= 1 \\ x+3y &= 1 \end{cases}$

Video : Example 9

Workouts

Exercise 1.
Compute :

1. $\begin{pmatrix} 1 & 2 & -3 \\ 0 & -4 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 5 & 1 \\ 1 & -2 & 0 \end{pmatrix}$
2. $-3\begin{pmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{pmatrix}$

Exercise 2.
Compute AB and BA with :
$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix}$ et $B = \begin{pmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{pmatrix}$

Exercise 3.
Let $A = \begin{pmatrix} a & -a \\ 1 & a+1 \end{pmatrix}$ Determine a so that the matrix A is invertible. then compute A^{-1}.

Exercise 4.
Find two matrices A and B satisfying $AB = BA$.

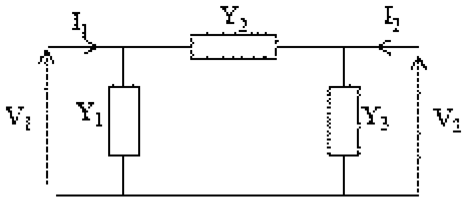
Exercise 5.
Solve the following systems using matrix algebra :

1. $\begin{cases} x+z=1 \\ -x-z=1 \end{cases}$
2. $\begin{cases} ax+y=b \\ x+2y=1 \end{cases}$

Exercise 6.
Two students took tennis lessons. One took 16 hours of lessons and made three internship : she paid 558 euros, the other took 18h lessons has two internships and paid 460 euros. Translate the statement above as a system, then under matrix form.
Deduce the price of a lesson and an internship.

Exercise 7.

Consider the following circuit :



We get this system :

$$\begin{cases} \underline{I_1} = (\underline{Y_1} + \underline{Y_2})\underline{V_1} - \underline{Y_2}\,\underline{V_2} \\ \underline{I_2} = -\underline{Y_2}\,\underline{V_1} + (\underline{Y_3} + \underline{Y_2})\underline{V_2} \end{cases}$$

Translate this system in matrix form and deduce the vector $\begin{bmatrix} \underline{V_1} \\ \underline{V_2} \end{bmatrix}$ in function of $\begin{bmatrix} \underline{I_1} \\ \underline{I_2} \end{bmatrix}$.

Exercise 8.
Let A and B two two by two matrices. Show that $\det AB = \det A \times \det B$.

Exercise 9.
Let A, B and C three square matrices, and B an invertible matrix such that : $A = BCB^{-1}$. Express C based on A and B.

Exercise 10.
let the matrices

$$A = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix} ; J = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} ; I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1. Find the real numbers a and b such that $A = aI + bJ$
2. Compute J^2
3. Compute A^2, A^3 and A^4 as a linear combination of matrices I and J.