VECTOR SPACES

Objectives
- **Understand the notion of Vector Space.**
- **Subspaces.**
- **Understand and find Basis.**

In this chapter, we use in a generic way a set $\mathbb{K}$ which represents either $\mathbb{R}$ or $\mathbb{C}$.
$E_1 \times E_2 \cdots \times E_n = \{(x_1, x_2, \ldots, x_n) \text{ such that } x_i \in E_i\}$.

Example 1.
Describe $\mathbb{R}^3$.

1 Vector spaces

1.1 Groups

Let E be a set equipped with a binary operation $\oplus$ that combines any two elements of E.

Example 2.
In each case below, take two elements u and v of E, and compute $u \oplus v$.
- $E = \mathbb{R}^2$ and $\oplus$ is the common addition on $\mathbb{R}^2$.
- $E = \mathbb{R}^2$ and $(x, y) \oplus (x', y') = (x + y', x' + y)$.
- $E = \mathbb{R}^2$ and $\oplus$ is the dot or scalar product.
- $E = \mathbb{R}$ and $u \oplus v = u \times v + (u^2 - 1)(v^2 - 1)$

Video : example

A group, denoted $(E, \oplus)$), is an algebraic structure consisting of a set of elements E equipped with an operation $\oplus$ that combines any two elements to form a third element. (The operation satisfies five conditions called the group axioms, namely closure, associativity, commutatitvity, identity and invertibility.)

(A0)
$$\forall (u, v) \in E^2, u \oplus v \in E$$
closure.

(A1)
$$\forall (u, v) \in E^2, u \oplus v = v \oplus u$$
$\oplus$ is **commutative**.

(A2)
$$\forall (u, v, w) \in E^3, (u \oplus v) \oplus w = u \oplus (v \oplus w)$$
$\oplus$ is **associative**

(A3)

iii There exists an element, such an element is unique called **the identity element** for $\oplus$, denoted by 0_E called the zero vector, such that

$$\forall u \in E, 0_E \oplus u = u$$

(A4) For each element of E, there exists an element, commonly denoted by $-u$ such that

$$\forall u \in E, u \oplus (-u) = 0_E$$

. It is called the inverse element. Instead of writting $u + (-u) = 0_E$ on we could write : $u - u = 0_E$

Example 3.
In each previous example, check if $(E, \oplus)$ is or not a commutative group :
Video : closure
Video : commutativity
Do on your own associativity and identity element.

1.2 Vector Space

Every number belonging to $\mathbb{K}$ is called a scalar.

Let E be a set endowed with an operation denoted by $\oplus$ and called vector addition or simply addition. The second operation, called scalar multiplication takes any scalar λ and any vector $u \in E$ and gives another vector $\lambda \odot u$.

E is endowed with two operations $\oplus$ et $\odot$.

E together with those two operations $\oplus$ and $\odot$, (denoted by $(E, \oplus, \odot)$) is a vector space over the field $\mathbb{K}$ if $(E, \oplus, \odot)$ checks the six following conditions :

(A00)

$$(E, \oplus) \text{ is a commutative group.}$$

(M0)

$$\forall \alpha \in \mathbb{K}, \forall u \in E, \alpha \odot u \in E$$

the $\odot$ law is said to be external. This law is called external because we multiply a scalar by an element of E.

(M1) Distributivity of scalar multiplication with respect to vector addition :

$$\forall \alpha \in \mathbb{K}, \forall (u, v) \in E^2, \alpha \odot (u \oplus v) = (\alpha \odot u) \oplus (\alpha \odot v)$$

(M2) Distributivity of scalar multiplication with respect to field addition

$$\forall (\alpha, \beta) \in \mathbb{K}^2, \forall u \in E, (\alpha + \beta) \odot u = (\alpha \odot u) \oplus (\beta \odot u)$$

(M3) Compatibility of scalar multiplication with field multiplication

$$\forall (\alpha, \beta) \in \mathbb{K}^2, \forall u \in E, (\alpha\beta) \odot u = \alpha \odot (\beta \odot u)$$

(M4) Identity element of scalar multiplication :

$$\forall u \in E, 1 \odot u = u$$

Remark 1. The usual addition in $\mathbb{R}^n$ is denoted $+$. The usual scalar multiplication in $\mathbb{R}^n$ is denoted by ..

Example 4.

1. Prove that $(\mathbb{R}^2, +, .)$ is a vector space.
   Video : example 4)1)
2. Prove that $(\mathbb{R}^2, +, \odot)$ is not a vector space with $\lambda \in \mathbb{R}$, $\lambda(x, y) = (x + \lambda, y + \lambda)$.
   Video : example 4)2)

Remark 2. Please note that the vector space structure ie operations which we endow the set E, can make it or not, a vector space, as shown in the example above.

Definition 1.
Elements of the vector space E are called **vectors** and elements of $\mathbb{K}$ are called **scalar**.
In $\mathbb{R}^n$, we use the notation with an arrow but we won't use it as vectors can be functions.

Proposition 1.
$\mathbb{R}^n$ endowed with the common addition $+$ and the common scalar multiplication $\cdot$ is a vector space over $\mathbb{R}$.

Proposition 2.

1. $\forall u \in E,\ 0 \odot u = 0_E$
2. $\forall \lambda \in \mathbb{K},\ \lambda \odot 0_E = 0_E$
3. For all $u \in E$ and for all $\lambda \in \mathbb{K}$, $\lambda \odot u = 0_E \Rightarrow \lambda = 0 \text{ ou } u = 0_E$
4. For all $u \in E$, $(-1) \odot u = -u$

Example 5.
Prove the previous proposition (1,2 and 3).
Video : 1
Video : 2
Video : 3

2 Subspaces

Definition 2.
Let F be a part of a vector space $(E, +, \cdot)$ over $\mathbb{K}$
We say that F is a subspace of the vector space E
— F is non empty
— F endowed with the two operations $+$ and $\cdot$ of E is itself a vector space.

This definition is not very useful as to prove that F is a subspace we have to prove that F is itself a vector space. Let's introduce another interesting theorem.

Theorem 3. Let F be a subset of a vector space $(E, +, \cdot)$over $\mathbb{K}$
F is a subspace of E if it checks those three conditions :
(i) $0_E \in F$
(ii) $\forall (u, v) \in F^2, u + v \in F$. If we add any two vectors we end up with a vector of F
(iii) $\forall u \in F, \forall \alpha \in \mathbb{K}, \alpha \cdot u \in F$. If we multiply any vector by a constant we end up with a vector of F.

Example 6. Prove this theorem.
Video : ex 6

Using this theorem, it will be easier to prove that a subset F of E is a subspace. There exists another summary version of this theorem

Theorem 4. Let F be a subset of a vector space $(E, +, \cdot)$over $\mathbb{K}$
F is a subspace of E if it checks those two conditions :
(i) $0_E \in F$
(ii) $\forall (u, v) \in F^2, \forall (\alpha) \in \mathbb{K}, \alpha \cdot u + v \in F$
This is a summary version. You choose either this theorem or the previous one.

Example 7.
Show that the plane of equation $2x - 3y + 2z = 0$ is a vector subspace of $\mathbb{R}^3$.
Video : ex 7
On your own : get training with exercise 3)

3 Subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

3.1 In $\mathbb{R}^2$

Video : subspaces of $\mathbb{R}^2$

Remark 3.
$\mathbb{R}^2$ can be viewed as a set of **points** M of coordinates (x, y) in the xy-plane $(O, \vec{i}, \vec{j})$. But in this chapter $\mathbb{R}^2$ is seen as a set of **vectors** $\vec{u}$ whose coordinates are (x, y) in the standard basis $(\vec{i}, \vec{j})$.

Proposition 5.
Let $\vec{u} = (a, b)$ be the direction vector of a straight line D trough the origin, in a basis $(\vec{i}, \vec{j})$

1. D has a cartesian equation : $-bx + ay = 0$.
2. $D = \{(\lambda a, \lambda b),\ \lambda \in \mathbb{R}\}$.
3. $D = \mathcal{S}pan(\vec{u})$

Proposition 6. Subspaces of $\mathbb{R}^2$
Subspaces of $\mathbb{R}^2$ are $\{0_E\}$, the straight lines trough the origin and $\mathbb{R}^2$.

3.2 In $\mathbb{R}^3$

Video : subspaces of $\mathbb{R}^3$

Remark 4.
$\mathbb{R}^3$ can be viewed as a set of **points** M of coordinates (x, y, z) in $(O, \vec{i}, \vec{j}, \vec{k})$. But in this chapter $\mathbb{R}^3$ is seen as a set of **vectors** $\vec{u}$ whose coordinates are (x, y, z) in th standard basis $(\vec{i}, \vec{j}, \vec{k})$.

Proposition 7. Plane through the origin
Let $\vec{u} = (a, b, c)$ and $\vec{v} = (a', b', c')$ be two non colinear vectors, in a basis $(\vec{i}, \vec{j}, \vec{k})$, the plane trough the origin P spanned by those two vectors

1. has a cartesian equation : $(bc' - cb')x + (ca' - ac')y + (ab' - a'b)z = 0$
2. $P = \{(\lambda a + \mu a', \lambda b + \mu b', \lambda c + \mu c'),\ \lambda \in \mathbb{R} \quad \mu \in \mathbb{R}\}$.
3. $P = \mathcal{S}pan(\vec{u}, \vec{v})$

Proposition 8. Straight line trough the origin
Let $\vec{u} = (a, b, c)$ be the direction vector of a straight line D trough the origin, in a basis $(\vec{i}, \vec{j}, \vec{k})$

1. D has a cartesian equation : $-bx + ay = 0$and $\quad - bz + cy = 0$
2. $D = \{(\lambda a, \lambda b, \lambda c),\ \lambda \in \mathbb{R}\}$.
3. $D = \mathcal{S}pan(\vec{u})$

Proposition 9. subspaces of $\mathbb{R}^3$
Video : subspaces of $\mathbb{R}^3$
Subspaces of $\mathbb{R}^3$ are $\{0_E\}$, vector lines, vector planes and $\mathbb{R}^3$.

Remark 5.
In $\mathbb{R}^3$ two vector lines are always coplanar. Parallelism does not make sense.

Example 8.
Give the system of equations of the line of $\mathbb{R}^3$ spanned by the vector $(2, -1, 3)$.
Video : ex 8

4 Intersection of subspaces

Proposition 10.
The intersection of two subspaces F and G of a vector space E is a vector space itself.

However the union of two subspaces is not in general a vector subspace.

Generally, let $(E, +, \cdot)$ be a vector space over $\mathbb{K}$, let I be a non empty set and $(F_i)_{i \in I}$ a familly of subspaces of E. The intersection $F = \bigcap_{i \in I} F_i$ is a subspace of E.

Example 9.

1. Check on your own that in $\mathbb{R}^3$,the intersection of two subspaces of a vectoriel space is a subspace.
2. Prove that the intersection of two subspaces of a vector space is a subspace. Video : ex 92)
3. Prove that the union of two subspaces of a vector space is not a subspace in general. Video : ex 93)

5 Sum of subspaces

5.1 Defintion and properties

Definition 3.
Let E be a vector space over $\mathbb{K}$, let F and G be two subspaces of E. We can perform sum operation, denoted by $F+G$, this is the set of vectors which are the sum of a vector of F and a vector of G :

$$F+G=\{u\in E/u=f+g, f\in F, g\in G\}$$

Remark 6. Every element of $F+G$ is a sum of an element of F and an element of G, which means $u\in F+G\Leftrightarrow \exists f\in F, \exists g\in G$ tels que $u=f+g$. This way of writting is not unique generally.

Example 10.
Let D and D' be two straight lines trough the origin of $\mathbb{R}^3$. Find $D+D'$.
Video : ex 10

Theorem 11.
The sum of two sub vector spaces of a vector space E is a subspace of E.

Example 11.
Prove this theorem.
Video : ex 11

Remark 7.
Be careful not to be confused with ths sum notation $+$ and avoid mistakes :

1. $F+F=F$
2. By setting $-F=\{-x, x\in F\}$, we get $-F=F$
3. If $F\subset G$, $F+G=G+G$ even though $F\neq G$

5.2 Direct sum

Definition 4.
et E be a vector space over $\mathbb{K}$, F and G two subspaces of E. The sum $F+G$ is direct if every vector of $F+G$ has a unique expression as a sum of an element of F and an element of G.
If the sum between F and G is dierct, we use this notation $F+G=F\oplus G$

Theorem 12.
Let E be a vector space over $\mathbb{K}$, F and G two subspaces of E.
Then : $F+G$ is direct $\Leftrightarrow F\cap G=\{0_E\}$

Example 12.
Prove this theorem.
Video : ex 12 part 1
Video : ex 12 part 2

Example 13.
For the following straight lines and planes trough the origin find $F+G$ and precise if the sum is direct or not.

5.3 Complementary subspaces

Definition 5.
Let E be a vector-space over $\mathbb{K}$, F and G two subspaces of E. F et G dare called complementary subspaces in E if $F+G$ is direct and equal to E. Thus F and G are complements in $E \Leftrightarrow E = F \oplus G$.

We say that G is a complement of F.

Two subspaces F and G of a vector space over $\mathbb{K}$ are complementary subspaces in E if and only if

$$F \cap G = \{0_E\} \quad F + G = E$$

Theorem 13.
Every vector subspace of E has a complement.

Remark 8.

1. A subspace F of E may have several complements. Let $\mathbb{K} = \mathbb{R}$ and $E = \mathbb{R}^2$, the subspace $F = \mathbb{R} \times \{0\}$ de E has infinitely many complementary subspaces in E, of the shape $\mathbb{R}x$ with $x \in E - F$: $F = Vect((1,0))$ then $D = Vect(2,1)$ is a complement of F in R^2 and so is $D' = Vect(1,0)$
2. In finite dimension, all subspace has at least one complementary subspace.
3. The existence of a complementary subspace in a vector space is equivalent to the axiom of choice

Theorem 14.
Let F and G be two subspaces of a vector space E. Then F and G are complements in E if and only if all vector $u \in E$ has a unique expression $u = f + g$ $f \in F$ and $g \in G$. Every element of $F + G$ has a unique expression as an element of F and an element of G.

Remark 9. Be careful, two subspaces may be complementary subspaces in a vector space but not in another one. For instance two straight lines trough the origin of $\mathbb{R}^3$ are complements in the half plane they span but not in the whole space $\mathbb{R}^3$, as even their sum is direct in $\mathbb{R}^3$ their direct sum is not $\mathbb{R}^3$.

Example 14. Let's consider $E = \mathbb{R}^3$. Prove that $F = \left\{(x, y, z) \in \mathbb{R}^3 / x - y + z = 0\right\}$ and $G = \left\{(x, x, x) \in \mathbb{R}^3\right\}$ are complements in E.
Video : ex 14 part 1
Video : ex 14 part 2

6 Finite vector families

6.1 Spanning family

Definition 6.
Let E be a vector space and $u_1, \ldots, u_n$, n vectors of E.

A vector u de E is a **linear combination** of $u_1, \ldots, u_n$, if there exists n scalars $\alpha_1, ..., \alpha_n$ of $\mathbb{K}$ such that

$$u = \alpha_1 u_1 + \cdots + \alpha_n u_n$$

.

Definition 7.
Let E be a vector space over $\mathbb{K}$l. The set of vectors $u_1, \dots, u_n$ is a **spanning family** of E if E is the set of all linear combinations of $u_1, u_2, \dots, u_n$. E is called the vector space spanned by $u_1, \dots, u_n$, and we denote it $E = \mathcal{S}pan(u_1, \dots, u_n)$.

$$u \in \mathcal{S}pan(u_1, u_2, \dots, u_n) \Leftrightarrow \exists(\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$$
$$u = \alpha_1 u_1 + \dots + \alpha_n u_n$$

Example 15. Let u and v be two vectors of $\mathbb{R}^3$. What can you say about $\mathcal{V}ect(u, v)$?
Video : ex 15

Example 16. Find two spanning families of the subspace E of $\mathbb{R}^3$ where E is the set of vectors $u = (x, y, z)$ such that : $x - y + z = 0$.
Video : ex 16

Theorem 15. Let E be a vector-space over $\mathbb{K}$ and $\mathcal{F} = \{u_1, u_2, u_i \dots, u_j, u_n\}$ a spanning family of E. The following families are also spanning families of E :

1. The family get buy switching two vectors of $\mathcal{F}$
2. The family get by multipliying one vector of $\mathcal{F}$ by a non zeo scalar.
3. The family get by adding to one vector of $\mathcal{F}$ a linear combination of other vectors of $\mathcal{F}$.
4. The family get by removing in $\mathcal{F}$ a vector which is a a linear combination of other vectors of $\mathcal{F}$.

Example 17.
Write the previous theorem in mathematics language.
Video : ex 17

Proposition 16.
If $F = \mathcal{V}ect\{u_1, u_2, ..., u_n\}$ et $G = \mathcal{V}ect\{v_1, v_2, ..., v_p\}$, then $F + G = \mathcal{V}ect\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_p\}$

Example 18.
Let u_1, u_2, u_3 be three vectors in a vector space E. What is $\mathcal{S}pan(u_1, u_2) + \mathcal{S}pan(u_3)$?
Video : ex 18

6.2 Linearly indepedence

Definition 8.
Let $\mathcal{F} = (u_1, u_2, \dots, u_n)$ be a family of vectors in a vector space E. We say that this family **is linearly independent** or that the vectors $u_1, u_2, \dots, u_n$ are **linearly independent**, if and only if a linear combination of those vectors which is zero implies that all coefficients are zero.Which means :

$$\forall(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{K}^n, \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_n u_n = 0 \Rightarrow$$
$$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

Example 19.

1. In the vector space $\mathbb{R}^3$ over $\mathbb{R}$, prove that the family $((1,2,0),(0,1,2))$ is linearly independent.
2. In the vector space of polynomials with real coefficients over $\mathbb{R}$, prove that the family $1, X, X-1$ is not linearly independent.

Video : ex 19

Remark 10.
Every sub-family of a linearly independent family is linear independent.

6.3 Linearly dependence

Definition 9.
Let $\mathcal{F} = (u_1, u_2, \ldots, u_n)$ be a family of vectors in a vector space E. This family is **linearly dependent** or the vectors $u_1, u_2, \ldots, u_n$ are **linearly dependent**,if and only if it is not linearlyindependent. Which means : $\exists(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{K}^n$ non all zero such that $\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n = 0$

Particular cases

1. If $n = 1$ then the set (u_1) is linearly dependent if $u_1 = 0$.
2. If $n = 2$ then the set (u_1, u_2) is linearly dependent iif u_1 et u_2 are collinear.
3. If $n = 3$ then the set (u_1, u_2, u_3) is linearly dependent iif u_1, u_2 et u_3 are coplanar.

Theorem 17.
A family $\mathcal{F} = (u_1, u_2, \ldots, u_n)$ is linearly dependent if one of those vectors is a linear combination of the others

6.4 Basis

Definition 10.
A family $\mathcal{F} = (e_1, \ldots, e_n)$ of vectors in a vector space E is a basis of E if and only if this family is both a **spannig family** of E and **linearly independent**.

Definition 11.
A **standard basis** of a vector space E is a very simple basis. We speak about the canonical basis.

Example 20.

— In $\mathbb{R}^2$, $e_1 = (0,1)$ $e_2 = (1,0)$ is the standard basis

— In the set of polynomials of degree less or equal than 2, $(1, X, X^2)$ is the standard basis.

Theorem 18.
Let $\mathcal{B} = (e_1, \ldots, e_n)$ be a basis of a vector space E over $\mathbb{K}$, u any vector of E. There exists a unique family $(x_1, \ldots, x_n) \in \mathbb{K}^n$ such that : $u = x_1 e_1 + \cdots + x_n e_n$.

Those coefficients $(x_1, \ldots, x_n)$ are the coordinates of u in the basis $\mathcal{B} = (e_1, \ldots, e_n)$. It is unique.

Particular cases

Let $\mathcal{F} = (e_1, \ldots, e_n)$ be a basis of E.

1. If $n = 1$ then E is a straight line trough zero
2. If $n = 2$ then E is a plane trough zero.

6.5 Spanning and Linearly independant families

Theorem 19. The Exchange Lemma
Let E be a vector-space over K and let $\mathcal{G} = \{x_1, x_2, ..., x_p\}$ be a spanning vector of E, and $\mathcal{L} = \{y_1, y_2, ..., y_r\}$ be a linearly independent family of E then :

$$r \leqslant p$$

there exists one way to replace r des vectors of $\mathcal{G}$ by vectors of $\mathcal{L}$.

7 Finite dimension vector space

7.1 Definitions and properties

Definition 12.
Let E be a vector space over $\mathbb{K}$. E is of finite dimension if and only if E has a finite basis.

Theorem 20. Dimension theorem
In a non zero vector space E over $\mathbb{K}$ of finite dimension all bases of a vector space have equally many elements. This finite number of elementsd defines the dimension of the space E and is denoted $\dim E$.

By convention $\{0_E\}$ has for dimension zero.

Example 21.
Are those subspaces finite or not ? If finite, give their dimension.

1. $\mathbb{R}^2$
2. A plane trough the origin.
3. The set of continuous functions on an interval.

Proposition 21. $\mathbb{R}^n$ is a vector space of dimension n over $\mathbb{R}$

Remark 11. The dimension of a vector space depends on the $\mathbb{K}$ on which we are working.

Theorem 22. Basis adapted to a direct sum
Let F and G be two vector subspaces of the $\mathbb{K}$ vector space E.
We give a $\mathcal{B} = (f_1, \dots, f_p)$ basis of F and $\mathcal{B}' = (g_1, \dots, g_q)$ a basis of G. So :

1. $F \cap G = \{0\} \Leftrightarrow$ the set $(f_1, \dots, f_p, g_1, \dots, g_q)$ is linearly independant in E.
2. $F + G = E \Leftrightarrow$ the set $(f_1, \dots, f_p, g_1, \dots, g_q)$ spans E.
3. $F \oplus G = E \Leftrightarrow$ the set $(f_1, \dots, f_p, g_1, \dots, g_q)$ is a basis of E.

Example 22.
Prove the above theorem.

Theorem 23. Incomplete basis theorem
Let E be a vector space over $\mathbb{K}$ of ifnite dimension. Every family of vectors of E linearly independent is a sub-family of a basis of E.

We are able to add suitably choosen vectors to a linearly independent family to get a basis of E.

This means that we can gradually add vectors to a suitably chosen linearly independant set in order to construct a base of E.

7.2 Dimension and cardinality

Theorem 24.
Let E be a vector space over $\mathbb{K}$, of finite dimension $n \in \mathbb{N}^*$. Let $\mathcal{F} = (u_1, \dots, u_p)$ be a family of vectors in E.

1. If $\mathcal{F}$ is linearly independent then $p \leqslant n$.
2. If $\mathcal{F}$ is a spanning family then $p \geqslant n$.
3. If $\mathcal{F}$ is either linearly independent or a spanning family and if $p = n$ then $\mathcal{F}$ is a basis of E.

Example 23. Justify the above theorem.

Corollary 25.

Let F and G be two vector subspaces of the vector space E such that $F \subset G$ and dim F=dim$G$ so $F = G$

Example 24.

Justify the above corollary.

8 Finite dimension vector subspaces

8.1 Dimension of a vector subspace

The notions of linearly independence and spanning family are the same as we place in E or in F vector subspace of the vector space E.

Theorem 26.
Every vector subspace F of a $\mathbb{K}$ vector space E of finite dimension is of finite dimension and we have : $\dim F \leqslant \dim E$

Theorem 27 (Grassmann Formula).
Let E be a vector space and F and G two vector subspaces of E then

$$\dim F + G = \dim F + \dim G - \dim F \cap G$$

Example 25.
Check the formula on the following examples.

8.2 Rank of a vector set

Definition 13.
Let E be a finite dimensional vector space.The rank of a family of vectors of E is the dimension of the subspace spanned by this family. We denote it by rg.

$$\mathrm{rg}(u_1, \dots, u_n) = \dim \mathcal{S}pan(u_1, \dots, u_n)$$

Example 26.
Let $u = (2, 3, 5), v = (4, 6, 10)$, et $w = (-2, -3, -5)$. Find $\text{rg}(u, v, w)$.

Theorem 28.
Let $(u_1, \ldots, u_p)$ be a family of vectors in a vector space E $\mathcal{K}$.
- If $\dim E = n$ then we get : $\text{rg}(u_1, \ldots, u_p) \leqslant n$
- $\text{rg}(u_1, \ldots, u_p) \leqslant p$
- $(u_1, \ldots, u_p)$ is linearly independent if and only if its rank is p.

8.3 Sub-spaces complements in finite dimension

Theorem 29.
Let E be a finite-dimensional vector space. Let F and G be two complement subspaces in E. So :

$$\dim F + \dim G = \dim E$$

Remark 12.
Be careful, the converse is false as the following counterexample shows : $F = Vect((1, 1))$ et $F = G$.

Theorem 30. Characterization theorem
Let E be a finite-dimensional vector space n.

1. If $\dim F + \dim G = \dim E$ and if $F \cap G = \{0\}$ then F and G are complements in E.
2. If $\dim F + \dim G = \dim E$ and if $F + G = E$ then F and G are complements in E.

Exercises

TD 1-3

Exercise 1.

1. Let $E = \mathbb{R}_+^* \times \mathbb{R}$. We define on E the addition by $(a,b) \oplus (c,d) = (ac, b+d)$ and the scalar law by $\lambda(a,b) = \left(a^\lambda, \lambda b\right)$. Show that $(E,+,.)$ is a $\mathbb{R}$ vector space.
2. On $E = \mathbb{R}^2$, we define the following operations $(a,b) \oplus (c,d) = (a+c, b+d)$ and $\lambda \odot (a,b) = (\lambda a, 0)$. Show that E with those two operations is not a $\mathbb{R}$ vector space.
3. The set of the real bijective functions from $\mathbb{R}$ in $\mathbb{R}$ endowded with the internal law $\circ$ and the multiplication by a scalar is a vector space on $\mathbb{R}$?

Exercise 2.
Show that the set of continuous functions with the usual operation $+$ and $\cdot$ on an interval I $\subset \mathbb{R}$ is a vector space on $\mathbb{R}$

Exercise 3.
Let $E = \mathbb{R}^3$. Are the following sub-sets vector subspaces of E ?

1. The set of triplets $(x;y;z)$ such that $x+y=0$.
2. The set of triplets $(x;y;z)$ such that $x=0$ ou $y=0$.
3. The set of triplets $(x;y;z)$ such that $x^2+y^2+z^2=10$.

Exercise 4.
We denote by E the vector space of real-valued functions functions (from $\mathbb{R}$ to $\mathbb{R}$), equipped with the addition and the multiplication by a real number. Are the following subsets subspaces of E ?

1. The set of polynomials of the second degree.
2. The set of functions such that $f(1) = 2f(0)$.
3. The set of functions such that $f(1) - f(0) = 1$.
4. The functions such that, $a \in \mathbb{R}$ being set, $f(x) = f(a-x)$ for all $x \in \mathbb{R}$.
5. The set of differentiable functions over an interval I.
6. The set of solutions of a first order linear differential equation.
7. The set of polynomials of degrees less than or equal to n.

Exercise 5.
Indicate without calculation the nature of the following sets :

1. $E_1 = \{(x,y) \in \mathbb{R}^2 | x-2y=0\}$.
2. $E_2 = \{(3\lambda, -\lambda) | \lambda \in \mathbb{R}\}$
3. $E_3 = \{(x,y,z) \in \mathbb{R}^3 | x-2y=0\}$
4. $E_4 = \{(x,y,z) \in \mathbb{R}^3 | x-2y=0 \text{ and } y+z=0\}$
5. $E_4 = \{(x,y,z) \in \mathbb{R}^3 | x-2y=0 \text{ or } y+z=0\}$
6. $E_5 = \{(3\lambda, -\lambda, 2\lambda) | \lambda \in \mathbb{R}\}$

Exercise 6. (Optional)
Let E be a $\mathbb{K}$ vector space and F and G two subspaces Vector of E.
Show that $F \cup G$ is a vector subspace of E if and only if $F \subset G$ or $G \subset F$.

TD 4-5

Exercise 7.
Let F and G be two vector spaces of a vector space E.

1. What about $a \in E$ if $F \cap G = \{a\}$.
2. What about F and G if $F \cup G = E$.

Exercise 8.
Let $F = f : \mathbb{R} \to \mathbb{R}, f(x) = ax + b, a \in \mathbb{R}, b \in \mathbb{R}$ and
$G = f : \mathbb{R} \to \mathbb{R}, f(x) = ax^2 + bx, a \in \mathbb{R}, b \in \mathbb{R}$

1. Show that F and G are vector subspaces of the vector space of continuous functions.
2. Find $F \cap G$ and verify that $F \cap G$ is a vector space.
3. Find $F \cup G$. Is $F \cup G$ a vector space ?
4. Same question with $F = f : \mathbb{R} \to \mathbb{R}, f(x) = ax + b, a \in \mathbb{R}, b \in \mathbb{R}$ and
   $G = f : \mathbb{R} \to \mathbb{R}, f(x) = ax^2 + bx + c, a \in \mathbb{R}, b \in \mathbb{R}, b \in \mathbb{R}$

Exercise 9.
Is this set $(x; y; z)$ such that $x = 0$ and $2x + y = 0$ a vector subspace of E ?

Exercise 10.
Let $P = \{(x, y, z) \in \mathbb{R}^3 | x + y + z = 0\}$ et $D = \{(x, y, z) \in \mathbb{R}^3 | x = y = z\}$. We admit that P and D are subspaces of $\mathbb{R}^3$.

1. Determine $P \cap D$.
2. Let $\overrightarrow{k}$ a unitary vector of D and let $\overrightarrow{u}$ be a vector of $\mathbb{R}^3$.
   (a) Check that $\overrightarrow{u} - (\overrightarrow{k}.\overrightarrow{u})\overrightarrow{k}$ is in P
   (b) Deduce that P and D are complementary subspaces.

Exercise 11.
Let E be the set of applications of $\mathbb{R}$ in $\mathbb{R}$. Consider the sets :
$P = \{f \in E / f$ is an even function $\}$ et $I = \{f \in E / f$ is an odd function $\}$

1. Show that P and I are complementary subspaces of E
2. Give the decomposition in $P \oplus, I$ of the following functions $:x \mapsto e^x$; $x \mapsto (1 + x)^6$; $x \mapsto \sin(x)$.

Exercise 12.
Let F and G be two vector subspaces of a vector space E.

1. What can we say about F and G if $\forall x \in E, \exists (a, b) \in F \times G \mid x = a + b$.
2. What can we say about F and G if $\exists x \in E | \exists! (a, b) \in F \times G | \; x = a + b$

TD6

Exercise 13.
Let u and v be two vectors of a vector space E, compare the following sets :
$A = \mathcal{V}ect(u, v)$ $B = \mathcal{V}ect(-u, v)$ $C = \mathcal{V}ect(u + 2v, v)$ $D = \mathcal{V}ect(u)$ $D = \mathcal{V}ect(u) + \mathcal{V}ect(v)$

Exercise 14.
Let u, v be two vectors of a vector space E, put $w = u - 2v$.

1. Is (u, v, w) linearly independant ?
2. We suppose that u and v are non-collinear vectors. Is the family (u, v) linearly independant ?
3. We suppose that u, v and w are not collinear two by two. Is the family (u, v, w) linearly independant ?

Exercise 15.
Let E be a vector space and $\mathcal{B} = (e_1, e_2, e_3)$ a basis of E

1. $u = 2(e_3 - e_1) + 5e_2$, determine the coordinates of v in the $\mathcal{B}$ basis
2. $u(2, -3, 1)$, $v(1, 2, 3)$ $w(-1, -9, -8)$. Are the vectors u, v and w linearly independant ?
3. Same question for $u(1, -1, 1)$, $v(2, 1, 3)$ $w(-1, 2, 4)$.

Exercise 16.
$\forall u \in \mathbb{R}^2$, show that u is a linear combination of $(1, 1)$ and $(3, 1)$.

Exercise 17.
In $\mathbb{R}^3$ we consider the triplets : $a = (-1; 2; 1)$, $b = (0; 1; -1)$, $u = (1; 0; -3)$ and $v = (-2; 5; 1)$.

1. Determine x so that $(x; 1; 2)$ is in $\mathcal{V}ect\ (a, b)$.
2. Show that $\mathcal{V}ect\ (a, b) = \mathcal{V}ect\ (u, v)$.

TD7-8-9

Exercise 18.
Let $\mathcal{F} = (e_1, \ldots, e_n)$ and $F = \text{vect}(\mathcal{F})$.

1. Is $\mathcal{F}$ a basis of F ?
2. What necessary and sufficient condition must we have on $\mathcal{F}$ so that $\mathcal{F}$ is a basis of F ?

Exercise 19.
Do the following sets span E ?

- (1,1), (3,1) with $E = \mathbb{R}^2$
- (1,0,2), (1,2,1) with $E = \mathbb{R}^3$

Are the following sets linearly independant ?

- (1,1),(1,2) in $\mathbb{R}^2$
- (2, 3),(-6,9) in $\mathbb{R}^2$
- (1,3,1) , (1,3,0), (0,3,1) in $\mathbb{R}^3$
- (1,3), (-1,-2), (0,1) in $\mathbb{R}^2$

Exercise 20.
Let (u, v, w) ba abasis of a $\mathbb{R}$ vector space E. Among the following sets, which ones are spanning, linearly independant or basis of E.

1. $(u, u - 2v + w, -v + w)$
2. $(u - v, v - w, w - u)$

3. $(u, u - 2v + w)$

Exercise 21.

1. In $\mathbb{R}^3$ give an example of a linearly independant set, wich is not spanning E
2. In $\mathbb{R}^3$ give an exemple of a non linearly independant spanning set of E.

Exercise 22.
Show that $\mathbb{R}^2 = \mathcal{V}ect\,((0;4),(-1;2),(-1;-2))$.
Is the decomposition of an element of $\mathbb{R}^2$ unique ?

Exercise 23.
Consider the vectors of $\mathbb{R}^4$: $u = (1, -2, 4, 1)$ and $v = (1, 0, 0, 2)$.

1. Determine $\mathcal{V}ect\,(u, v)$.
2. Complete the set (u, v) adding two vectors of the canonical basis of $\mathbb{R}^4$ in order to have a basis of $\mathbb{R}^4$.

Exercise 24.
Are the following vector spaces finite or infinite ? Give dimension of vector spaces of finite dimension.

1. The subspace of $\mathbb{R}^3$ whose equation is $2x - 3y = 0$.
2. The solutions of an homogeneous second-order differential equation with constant coefficients.
3. Polynomials of degrees less than or equal to n.
4. The set of polynomials.

Exercise 25.

1. Find the dimension of $\mathbb{C}$ considered as a $\mathbb{R}$ vector space.
2. Find the dimension of $\mathbb{C}$ considered as a $\mathbb{C}$ vector space.

Exercise 26.
Let's consider the $\mathbb{R}$ vector space $E = \mathbb{R}^3$. In each case below, find a basis and an complementary subspace of the vector subspace F such that :

1. $F = Vect(\overrightarrow{u}, \overrightarrow{v})$ where $\overrightarrow{u} = (1, 1, 0)$ and $\overrightarrow{v} = (2, 1, 1)$.
2. $F = Vect(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ where $\overrightarrow{u} = (-1, 1, 0)$, $\overrightarrow{v} = (2, 0, 1)$ and $\overrightarrow{w} = (1, 1, 1)$.
3. $F = \{(x, y, z) \in \mathbb{R}^3 / x - 2y + 3z = 0\}$

Exercise 27.
Let F and G be two sets of $\mathbb{R}^3$ defined by : $F = \left\{(x, y, z) \in \mathbb{R}^3 / x + y + z = 0\right\}$ and $G = \{(\lambda, \lambda, \lambda)\, / \lambda \in \mathbb{R}\}$

1. Show that F and G are subspaces of $\mathbb{R}^3$ and give a basis for each one.
2. Show that F and G are complements.

Exercise 28.
Consider the vectors of $\mathbb{R}^4$: $v_1 = (2, 1, 3, 4)$, $v_2 = (0, 1, 0, 1)$, $v_3 = (2, 2, 3, 0)$ and $v_4 = (2, -1, 3, 7)$

Let E be a subspace of $\mathbb{R}^4$ spanned by : $(v_1,\ v_2,\ v_3,\ v_4)$.

1. Show that $(v_1,\ v_2,\ v_3)$ is a basis of E and give the coordinates of v_4 in that basis.
2. Determine a vector v_5 so that $(v_1,\ v_2,\ v_3,\ v_5)$ be a basis of $\mathbb{R}^4$.
3. Deduce a complement F of E in $\mathbb{R}^4$.

Exercise 29.
In $\mathbb{R}^4$ consider the following vectors $\overrightarrow{u} = (1,0,1,0)$, $\overrightarrow{v} = (0,1,-1,0)$, $\overrightarrow{w} = (1,1,1,1)$, $\overrightarrow{x} = (0,0,1,0)$ and $\overrightarrow{y} = (1,1,0,-1)$. Let $F = Vect(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$ and $G = Vect(\overrightarrow{x}, \overrightarrow{y})$.

Give the dimensions of $F, G, F+G, F \cap G$?

Exercise 30.
Determine the rank of the following set :

In $\mathbb{R}^4$, $F = \{\ v_1,\ v_2,\ v_3,\ v_4\ \}$ with $v_1 = (0,1,1,1)$, $v_2 = (1,0,1,1)$, $v_3 = (1,1,0,1)$ and $v_4 = (1,1,1,0)$

Exercise 31.
Determine, according to the value of x, the rank of the following set :
$x_1 = (1,x,-1)\,, x_2 = (x,1,x)\,, x_3 = (-1,x,1)$