DS 2 M 1.1 2024-2025

Exercice 1 :

1) $\operatorname{ch}^2(x) = \frac{1}{4}\left(e^x + e^{-x}\right)^2 = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$

$\operatorname{sh}^2(x) = \frac{1}{4}\left(e^x - e^{-x}\right)^2 = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$

donc $\operatorname{ch}^2(x) - \operatorname{sh}^2(x) = \frac{4}{4} = 1$

2)

a) f minorée et non majorée. Vrai $f(x) = e^x$

$\forall x \quad f(x) \geq 0$ minorée

et $\lim\limits_{+\infty} f(x) = +\infty$ non majorée.

b) Vrai : un minimum est un minorant "atteint".

c) $f(x) = \operatorname{sh} x \operatorname{th} x$

$f(-x) = \operatorname{sh}(-x) \operatorname{th}(-x) = (-\operatorname{sh} x)(-\operatorname{th} x)$

$= f(x)$ f est paire.

d) $f(x) = \left(\frac{1}{x}\right)^{x^2} = e^{x^2 \ln\left(\frac{1}{x}\right)}$ $\quad D_f = \mathbb{R}_+^*$

e) i) Vrai $\forall x \in \mathbb{R} \quad f(x) \geq -7$

ii) Vrai $\forall x \in \mathbb{R} \quad f(x) \leq 70$

iii) Faux $f(4) = 66 \geq 4$ 4 n'est pas un maximum.

Exercice 2 :

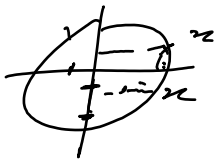
1) $f(x) = \cos 2x + \tan 3x$

$\cos 2x$: période $= \frac{2\pi}{2} = \pi$

$\tan 3x$: ——— $\frac{\pi}{3}$

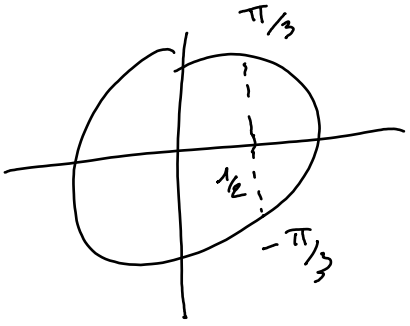
plus petite période commune : π

2) $\cos x = -\sin x = \cos\left(x + \frac{\pi}{2}\right)$



- donc $\begin{cases} x = x + \frac{\pi}{2} \quad [2\pi] \\ \text{ou} \\ x = -x - \frac{\pi}{2} \quad [2\pi] \end{cases}$ $\quad \begin{cases} 0 = \frac{\pi}{2} \ [2\pi] \text{ impossible} \\ 2x = -\frac{\pi}{2} \ [2\pi] \\ x = -\frac{\pi}{4} \ [\pi] \end{cases}$

3) $0 \leq \cos(2x + \pi) \leq \frac{1}{2}$



$\begin{cases} -\frac{\pi}{2} \leq 2x + \pi \leq -\frac{\pi}{3} \quad [2\pi] \\ \text{ou} \\ \pi/3 \leq 2x + \pi \leq \frac{\pi}{2} \quad [2\pi] \end{cases}$

$\begin{cases} -\frac{3\pi}{2} \leq 2x \leq -\frac{4\pi}{3} \ [2\pi] \\ -\frac{2\pi}{3} \leq 2x \leq -\frac{\pi}{2} \ [2\pi] \end{cases}$ $\quad \begin{cases} -\frac{3\pi}{4} \leq x \leq -\frac{2\pi}{3} \ [\pi] \\ -\frac{\pi}{3} \leq x \leq -\frac{\pi}{4} \ [\pi] \end{cases}$

Donc sur $[0; 2\pi[$: $\left[\frac{\pi}{4}; \frac{\pi}{3}\right]$ ou $\left[\frac{5\pi}{4}; \frac{4\pi}{3}\right]$

$\left[\frac{2\pi}{3}; \frac{3\pi}{4}\right]$ ou $\left[\frac{5\pi}{3}; \frac{7\pi}{4}\right]$

Exercice 3 :

1) (E) $3\frac{dU}{dt} + 5U = 5t^2 + 6t + 5 \qquad U(0) = 1$

Solution de l'équation homogène :

$$v_H = k e^{-\frac{5}{3}t}$$

Solution particulière :

sous la forme

$$v_p = at^2 + bt + c$$

$$\frac{dv_p}{dt} = 2at + b$$

$$5 \times \quad v_p = at^2 + bt + c$$

$$3 \times \quad \frac{dv_p}{dt} = 2at + b$$

$$= 5at^2 + (5b + 6a)t + 5c + 3b$$

par identification :

$$\begin{cases} 5a = 5 \\ 5b + 6a = 6 \\ 5c + 3b = 5 \end{cases} \Longrightarrow \begin{cases} a = 1 \\ b = 0 \\ c = 1 \end{cases}$$

Solution complète : $v = k e^{-\frac{5}{3}t} + t^2 + 1$

Condition Initiale : $v(0) = 1$

$$v(0) = 1 = k + 1 \qquad \underline{k = 0.}$$

Solution finale : $v(t) = t^2 + 1$

2)

- $f(x) = x \operatorname{sh}\left(\frac{x}{2}\right)$

$$f'(x) = x \cdot \frac{1}{2} \operatorname{ch}\left(\frac{x}{2}\right) + \operatorname{sh}\left(\frac{x}{2}\right)$$

$$f''(x) = x \cdot \frac{1}{4} \operatorname{sh}\left(\frac{x}{2}\right) + \operatorname{ch}\left(\frac{x}{2}\right)$$

$4f'' - f = 4\,\mathrm{ch}\left(\frac{x}{2}\right)$

- $4y'' - y = 4\,\mathrm{ch}\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)$

Solution de l'équation homogène :

$4r^2 - 1 = 0 \longrightarrow \alpha = -\frac{1}{2}\ ;\ \beta = \frac{1}{2}$

$y_H = Ae^{1/2\,x} + Be^{-1/2\,x}$

Solution particulière :

pour $4\,\mathrm{ch}\left(\frac{x}{2}\right)$ on a $y_p = x\,\mathrm{sh}\left(\frac{x}{2}\right)$

pour $\sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)$:

Sous la forme

$-1\times$	$y_p = a\sin\left(\frac{x}{2}\right) + b\cos\left(\frac{x}{2}\right)$
0	$y'_p = -\frac{1}{2}b\sin\left(\frac{x}{2}\right) + \frac{1}{2}a\cos\left(\frac{x}{2}\right)$
$4\times$	$y''_p = -\frac{1}{4}a\sin\left(\frac{x}{2}\right) - \frac{1}{4}b\cos\left(\frac{x}{2}\right)$

$= -2a\sin\left(\frac{x}{2}\right) - 2b\cos\left(\frac{x}{2}\right)$

donc par identification : $\begin{cases} -2a = 1 & a = -\frac{1}{2} \\ -2b = -1 & b = \frac{1}{2} \end{cases}$

Solution complète :

$y = Ae^{-1/2\,x} + Be^{1/2\,x} + x\,\mathrm{sh}\left(\frac{x}{2}\right) - \frac{1}{2}\sin\left(\frac{x}{2}\right) + \frac{1}{2}\cos\left(\frac{x}{2}\right)$

Exercice 4 :

$f(x) = |x|$

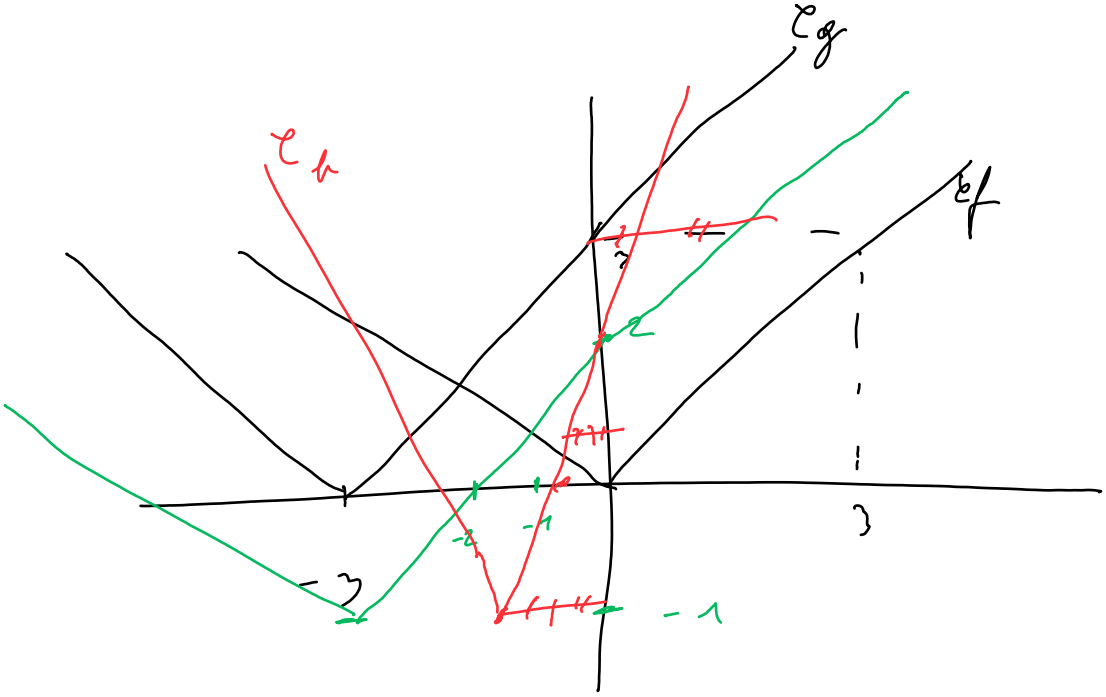
$g(x) = |x+3|$
$= f(x+3)$

de $\mathcal{C}_f$ à $\mathcal{C}_g$: translation à gauche de 3.

$h(x) = |2x+3| - 1$

$= g(2x) - 1$

de $\mathcal{C}_g$ à $\mathcal{C}_h$: dilatation horizontale de facteur $\frac{1}{2}$ et translation de 1 vers le bas.



Exercice 5 :

$m \geq 0$

$f(x) = \ln(x^2 + mx + 1)$

1) $\ln u$ est définie si $u > 0$.

signe de $x^2 + mx + 1$:

$\Delta = m^2 - 4$

3 cas : $\Delta > 0$ $\quad m^2 > 4$ $\quad m > 2$

dans le cas $\alpha = \dfrac{-m - \sqrt{m^2-4}}{2}$

$\beta = \dfrac{-m + \sqrt{m^2-4}}{2}$

x	$-\infty$		α		β		$+\infty$
x^2+mx+1		$+$	0	$-$	0	$+$	

et $Df =]-\infty ; \alpha[\cup]\beta ; +\infty[$.

$\Delta = 0$: $m = 2$.

$f(x) = \ln(x^2+2x+1) = \ln((x+1)^2)$

$Df = \mathbb{R} \setminus \{-1\}$

$\Delta < 0$: $\quad 0 \leq m < 2 \qquad x^2+mx+1 > 0$

et $Df = \mathbb{R}$.

2) $\lim\limits_{\pm\infty} x^2+mx+1 = \lim\limits_{\pm\infty} x^2 = +\infty$

donc $\lim\limits_{\pm\infty} f(x) = +\infty$.

cas 1 : $\lim\limits_{\alpha^-} f(x) = \lim\limits_{X \to 0^+} \ln X = -\infty$

$\lim\limits_{\beta^+} f(x) = -\infty$

cas 2 : $\lim_{-1} f(x) = -\infty$.

3] $\dfrac{f(x)}{x} = \dfrac{\ln(x^2 + mx + 1)}{x}$

$= \dfrac{\ln\left(x^2\left(1 + \frac{mx}{x^2} + \frac{1}{x^2}\right)\right)}{x}$

$= 2\dfrac{\ln x}{x} + \dfrac{\ln\left(1 + \frac{m}{x} + \frac{1}{x^2}\right)}{x}$

$\lim_{\pm\infty} \dfrac{\ln x}{x} = 0$ par croissance comparée.

$\lim_{\pm\infty} \dfrac{\ln\left(1 + \frac{m}{x} + \frac{1}{x^2}\right)}{x} = \text{"}\dfrac{0}{\pm\infty}\text{"} = 0$.

branche parabolique de direction Ox en $\pm\infty$.

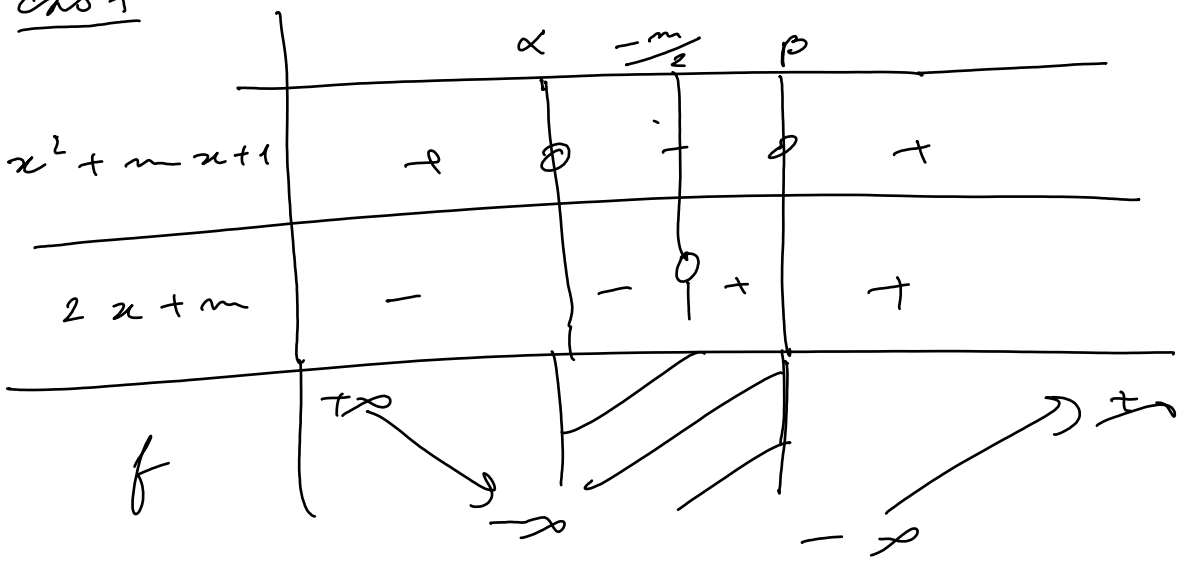
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cas 1 2 asymptotes Verticales en α et β.

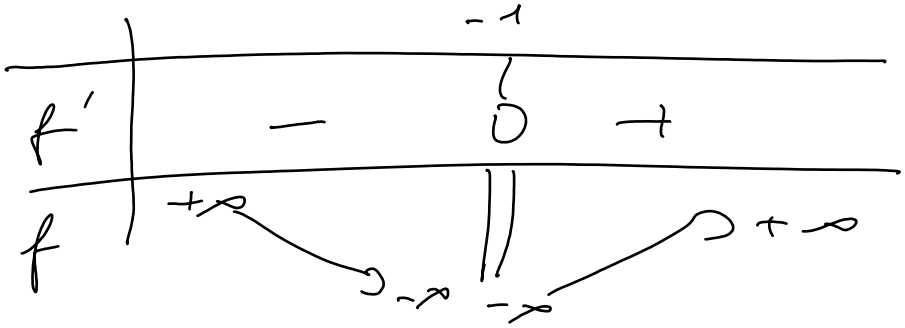
cas 2 1 ——————— $m - 1$

4] $f'(x) = \dfrac{2x + m}{x^2 + mx + 1}$

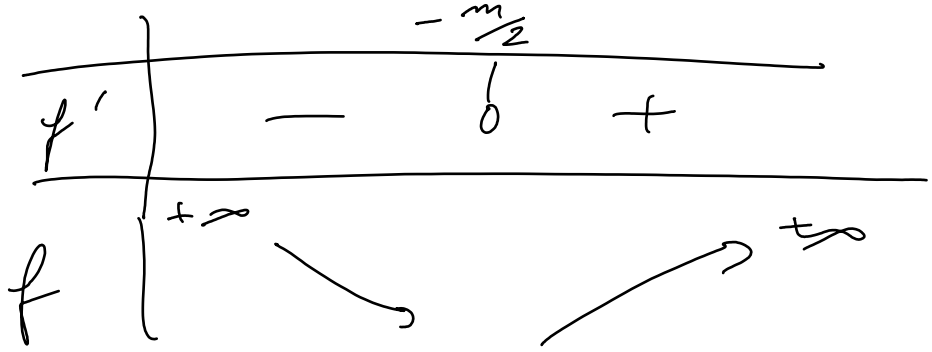
caso 1



caso 2:

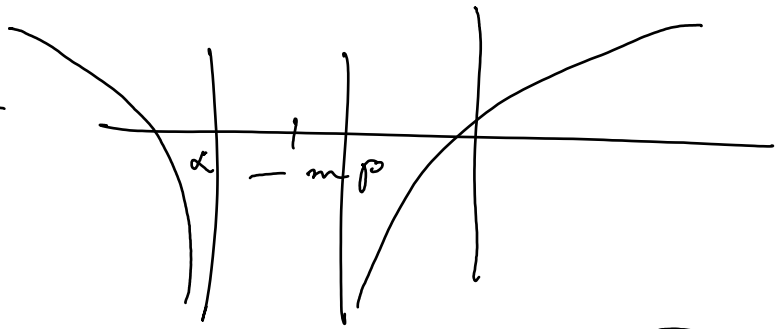


caso 3:

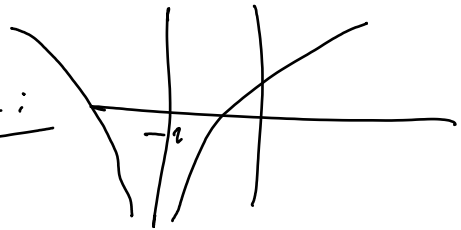


$f(-m/2)$

5] caso 1:



caso 2:



caso 3

