Exo1 :

$f \geq 0 \Rightarrow \inf_{[x_{i-1},x_i]} f \geq 0 \Rightarrow \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in [x_{i-1},x_i]} f \geq 0$

où (x_i) subdivision de $[a,b]$, donc par passage à la limite quand le pas de la subdivision tend vers $0 \Rightarrow$

$\int_a^b f \geqslant 0$

Exo2 :

$$\sum_{k=1}^{n} \frac{n}{n+k} \ln\left(\sqrt[n]{\frac{n+k}{n}}\right) = \sum_{k=1}^{n} \left(\frac{1}{1+k/n} \ln\left(1+\frac{k}{n}\right)\right) \cdot \frac{1}{n}$$

$b-a = 1$

$a = 0$

$f(x) = \frac{\ln(x+1)}{x+1} = u'u$ avec $u = \ln(x+1)$

car $\sum \frac{b-a}{n} f\left(a+k\frac{b-a}{n}\right) \xrightarrow[n\to\infty]{} \int_a^b f$

lorsque $n \to \infty$

$$\int_0^1 \frac{\ln(x+1)}{x+1}\,dx = \left[\frac{1}{2}u^2\right]_0^1$$

$$= \left[\frac{1}{2}\left(\ln(x+1)\right)^2\right]_0^1$$

$$= \frac{1}{2}(\ln 2)^2$$

Exo4 : $\int \frac{x^3}{\sqrt{1-4x^2}}\,dx = \int x^2 \left(\frac{x}{\sqrt{1-4x^2}}\,dx\right) =$

$u = \sqrt{1-4x^2} \Leftrightarrow u \geqslant 0 \quad x^2 = \frac{1-u^2}{4}$

$du = \frac{-4x}{\sqrt{1-x^2}}$

$$= \int \frac{(1-u^2)}{4}\left(\frac{-du}{4}\right) = \frac{1}{16}\left[u - \frac{u^3}{3} + C\right]$$

$$= -\frac{1}{16}\left[\sqrt{1-4x^2} - \frac{1}{3}\left(\sqrt{1-4x^2}\right)^3 + C\right]$$

$$\int x^2 \arcsin(2x)\,dx =$$

ibp

$u(x) = \arcsin(2x) \qquad v(x) = x^3/3$

$u'(x) = \dfrac{2}{\sqrt{1-4x^2}} \qquad v'(x) = x^2$

$$= \left[\frac{x^3}{3}\arcsin(2x)\right] - \int \frac{2}{3}\frac{x^3}{\sqrt{1-4x^2}}\,dx$$

$$= \frac{x^3}{3}\arcsin(2x) - \frac{2}{3}\left(-\frac{1}{16}\left(\sqrt{1-4x^2} - \frac{1}{3}\left(\sqrt{1-4x^2}\right)^3\right)\right)$$

$$= \frac{x^3}{3}\arcsin(2x) + \frac{1}{24}\left(\sqrt{1-4x^2} - \frac{1}{3}(1-4x^2)\sqrt{1-4x^2}\right)$$

$$= \frac{x^3}{3}\arcsin(2x) + \frac{1}{24}\sqrt{1-4x^2}\left(1 - \frac{1}{3} + 4x^2\right)$$

$$= \frac{x^3}{3}\arcsin(2x) + \frac{1}{3\times 24}\sqrt{1-4x^2}\left(2 + 4x^2\right)$$

$$= \frac{x^3}{3}\arcsin(2x) + \frac{1}{36}\sqrt{1-4x^2}\left(1 + 2x^2\right) + C$$

Exo 3 :

1) $\int t e^{-3t^2} dt = -\frac{1}{6} \int (-6t) e^{-3t^2} dt$

$= \frac{-1}{6} \int u'(t) e^{u(t)} dt$ avec $u(t) = e^{-3t^2}$

$= \frac{-1}{6} \left(e^{-3t^2} \right) + C$ $C \in \mathbb{R}$

2) $\int \cos^3 t \sin^2 t \, dt = \int \cos^2 t \cdot \cos t \cdot \sin^2 t \, dt$

$= \int (1 - \sin^2 t) \cos t \sin^2 t \, dt$

$= \int \cos t \sin^2 t \, dt - \int \cos t \sin^4 t \, dt$

$= \int u'(t) u^2(t) \, dt - \int u'(t) u^4(t) dt$ $u(t) = \sin t$, $u'(t) = \cos t$

$= \frac{u^3(t)}{3} - \frac{u^5(t)}{5} + C$

$= \frac{1}{3} (\sin t)^3 - \frac{1}{5} (\sin t)^5 + C$ $C \in \mathbb{R}$

3) $\int \ln(3x) dx \underset{\text{ipp}}{=} [x \ln(3x)] - \int 1 \, dx = x \ln(3x) - x + C$ $C \in \mathbb{R}$

$u(x) = \ln(3x)$ $v(x) = x$

$u'(x) = 1/x$ $v'(x) = 1$

4) $\int_1^e \frac{e^{2x}}{1 + e^x} dx = \int_1^e \frac{e^x}{1 + e^x} \cdot e^x dx$ $u = e^x$, $du = e^x dx$

$= \int_{e^1}^{e^e} \frac{u}{1+u} du$

$= \int_e^{e^e} \frac{u + 1 - 1}{1+u} du = \int_e^{e^e} 1 - \frac{1}{1+u} du$

$= \left[x - \ln|1+x| \right]_e^{e^e} = e^e - \ln(1+e^e) - e + \ln(1+e)$

5) $$\int \frac{2t + 3\sqrt{t}}{t^2}\,dt = \int \frac{2}{t} + 3t^{-3/2}\,dt$$

$$= 2\ln|t| + 3\,\frac{t^{-3/2+1}}{1-3/2} + C$$

$$= 2\ln|t| + 3(-2)\cdot\frac{1}{\sqrt{t}} + C$$

6) $$\int_0^{\pi/2} \cos^2(3t)\,dt = \int_0^{\pi/2} \frac{1+\cos(6t)}{2}\,dt$$

or $\cos^2 x + \sin^2 x = 1$

$\cos^2 x - \sin^2 x = \cos(2x)$

$\cos^2(x) = \frac{1+\cos(2x)}{2}$

$$= \left[\frac{1}{2}t + \frac{1}{2}\left(\frac{\sin(6t)}{6}\right)\right]_{x=0}^{x=\frac{\pi}{2}}$$

$$= \frac{\pi}{4} + \frac{1}{4}\sin(3\pi) - 0 - 0$$

$$= \pi/4$$

Exo 5:

$$\omega(t) = \frac{5\sin t}{\cos^2 t + 2}\,dt$$

$$\omega(-t) = \frac{-5\sin t\,(-dt)}{\cos^2 t + 2} = \omega(t)$$

on pose $u = \cos t$, $du = -\sin t\,dt$

$$I = \int_{\pi/6}^{\pi/3} \frac{5\sin t}{\cos^2 t + 2}\,dt = \int_{\pi/6}^{\pi/3} \frac{5\sin t}{\cos^2 t + 2}\cdot\frac{1}{-\sin t}\cdot(-\sin t\,dt)$$

$$= -5\int_{\pi/6}^{\pi/3} \frac{1}{\cos^2 t + 2}\,(-\sin t\,dt)$$

$t = \frac{\pi}{3} \Rightarrow u = \frac{1}{2}$

$t = \frac{\pi}{6} \Rightarrow u = \frac{\sqrt{3}}{2}$

$$= -5\int_{\sqrt{3}/2}^{1/2} \frac{1}{u^2+2}\,du$$

$$= 5\int_{1/2}^{\sqrt{3}/2} \frac{1}{2\left(\left(\frac{u}{\sqrt{2}}\right)^2+1\right)}\,du$$

$$= \frac{5}{2}\sqrt{2}\left[\arctan\left(\frac{u}{\sqrt{2}}\right)\right]_{1/2}^{\sqrt{3}/2}$$

$$= \frac{5\sqrt{2}}{2}\left(\arctan\left(\frac{\sqrt{3}}{2\sqrt{2}}\right) - \arctan\left(\frac{1}{2\sqrt{2}}\right)\right)$$

Exo 6 :

1) $f(x,y) = (x^2+y^2)\sin\left(\frac{1}{xy}\right)$

$D_f = \{(x,y) \in \mathbb{R}^2 / xy \neq 0\}$

$xy = 0 \iff x = 0$ ou $y = 0$

$D_f = \mathbb{R}^* \times \mathbb{R}^*$

$|f(x,y)| \leq x^2+y^2$

$\Rightarrow$ par les gendarmes

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

car $(x^2+y^2) \to 0$ quand $(x,y)\to(0,0)$

Ainsi f est prolongeable par continuité en posant $f(0,0) = 0$

$$\tilde{f} : (x,y) \to \begin{cases} f(x,y) & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases}$$

2) $f(x,y) = \dfrac{(x^2+y^2)\left(x - \sqrt{x^2+y^2}\right)}{y^2}$

$D_f = \{(x,y) \in \mathbb{R}^2 / y \neq 0\}$

$= \mathbb{R}^2 \setminus \{(x,0), x \in \mathbb{R}\}$

$= \mathbb{R} \times \mathbb{R}^*$

En utilisant les coordonnées polaires

$x = r\cos\theta \quad r > 0$

$y = r\sin\theta$

$f(r\cos\theta, r\sin\theta)$

$= \dfrac{r^2(r\cos\theta - r)}{r^2\sin^2\theta} = \dfrac{r(\cos\theta - 1)}{1-\cos^2\theta}$

$= \dfrac{-r}{1+\cos\theta} \quad \theta \to \pi$: FI

- $f(n, n^2) = \dfrac{(n^2+n^4)\left(n - \sqrt{n^2+n^4}\right)}{n^4} = \dfrac{(1+n^2)\left(1+\sqrt{1+n^2}\right)}{n}$

si $n < 0$: $n \to 0^-$; $\sqrt{n^2} = |n|$; car $|n| = -n$

$\Rightarrow \lim\limits_{0^-} f(n, n^2) = -\infty$

$\Rightarrow$ $\lim f$ en $(0,0)$ n'est pas finie

$\Rightarrow$ f n'est pas prolongeable par continuité en $(0,0)$

Exo 7 :

$f(x,y) = xe^{xy} + y^2$

$\dfrac{\partial f}{\partial x}(x,y) = 1 \cdot e^{xy} + xye^{xy}$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = yc^{xy} + y\left(e^{xy} + xyc^{xy}\right)$$
$$= 2yc^{xy} + xy^2 e^{xy}$$
$$= yc^{xy}(2 + xy)$$

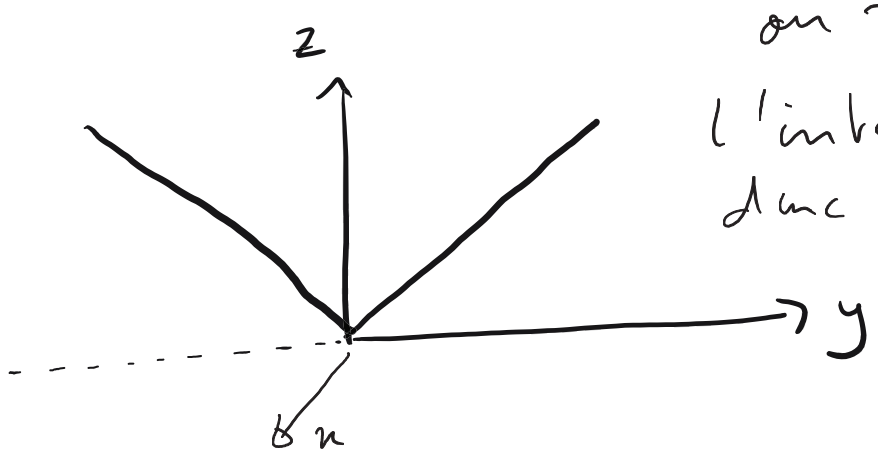
Exo 8 :

$$f(x,y) = \sqrt{x^2+y^2}$$

$M(x,y,z) \in S \cap (O,\vec{i},\vec{k}) \Leftrightarrow \begin{cases} x = 0 \\ z = \sqrt{x^2+y^2} \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ z = \sqrt{y^2} \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \\ z = y \text{ si } y \geq 0 \text{ ou } z = -y \text{ si } y \leq 0 \end{cases}$

ou $z = |y|$



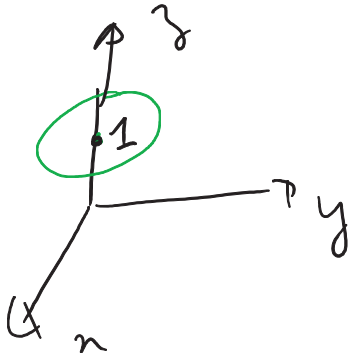
l'intersection est donc la réunion de 2 demi-droites.

2) $M(x,y,z) \in S \cap P_2 \Leftrightarrow \begin{cases} z = 1 \\ z = \sqrt{x^2+y^2} \end{cases}$

$\Leftrightarrow \begin{cases} z = 1 \\ \sqrt{x^2+y^2} = 1 \end{cases}$

$\Leftrightarrow \begin{cases} z = 1 \\ x^2+y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} z = 1 \\ M \in \mathcal{C}(0,1) \end{cases}$

Nature



$z = 1$