Exo 1 :

$$\frac{\partial f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = x^2 + y \quad (E)$$

$$\Leftrightarrow \quad \frac{\partial f}{\partial y} = \frac{x^3}{3} + yx + a(y) \quad a \ C^1$$

$$\Leftrightarrow \quad f(x,y) = \frac{x^3}{3}y + x\frac{y^2}{2} + \int a(y)\,dy + B(x)$$

$B \ C^2$

A primitive de a, $A \ C^2$

Exo 2 :

$$(x, y) \longrightarrow (u = x^2 + y, \ v = xy) \longrightarrow g(u,v) = f(x,y)$$

1) $$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial g}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial g}{\partial v}$$

$$\boxed{\frac{\partial f}{\partial x} = 2x\frac{\partial g}{\partial u} + y\frac{\partial g}{\partial v}} \qquad = 2x\frac{\partial g}{\partial u}(x^2+y, xy) + y\frac{\partial g}{\partial v}(x^2+y, xy)$$

2) $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial g}{\partial u} + y\frac{\partial g}{\partial v}\right)$$

$$= 2\cdot\frac{\partial g}{\partial u} + (2x)\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial u}\right) + 0 + y\cdot\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial v}\right)$$

$$= 2\frac{\partial g}{\partial u} + (2x)\left[\frac{\partial u}{\partial x}\frac{\partial^2 g}{\partial u^2} + \frac{\partial v}{\partial x}\frac{\partial^2 g}{\partial v \partial u}\right]$$

$$+ y\left[\frac{\partial u}{\partial x}\frac{\partial^2 g}{\partial u \partial v} + \frac{\partial v}{\partial x}\frac{\partial^2 g}{\partial v^2}\right]$$

$g \in C^2$ $\quad \dfrac{\partial^2 g}{\partial u \partial v} = \dfrac{\partial^2 g}{\partial v \partial u}$ par Schwartz

$$= 2\frac{\partial g}{\partial u} + (2x)\left[2x\frac{\partial^2 g}{\partial u^2} + y\frac{\partial^2 g}{\partial v \partial u}\right] + y\left[2x\frac{\partial^2 g}{\partial u \partial v} + y\frac{\partial^2 g}{\partial v^2}\right]$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial g}{\partial u} + 4x^2\frac{\partial^2 g}{\partial u^2} + y^2\frac{\partial^2 g}{\partial v^2} + 4xy\frac{\partial^2 g}{\partial u \partial v}}$$

Exo 3 :

1) $D_f = \{(x,y) \in \mathbb{R}^2 / |x|+|y| \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}$

2) a) $g(r\cos\theta, r\sin\theta) = \dfrac{r^3(\cos^3\theta + \sin^3\theta)}{r(|\cos\theta| + |\sin\theta|)}$

$= \dfrac{r^2(\cos^3\theta + \sin^3\theta)}{|\cos\theta| + |\sin\theta|} \xrightarrow{(*)} 0 = g(0,0)$

donc g est continue en $(0,0)$

$(*)$ car $|\cos\theta| + |\sin\theta| = \dfrac{2}{\sqrt{2}} \sin\left(\theta + \dfrac{\pi}{4}\right) \geq 1 \quad \forall \theta \in [0; \frac{\pi}{2}]$

donc $|\cos\theta| + |\sin\theta| \to 0 \quad \forall \theta$.

b)

$$\frac{\partial g}{\partial x}(0,0) = \lim_{t \to 0} \frac{g(0+t,0) - g(0,0)}{t} = \lim_{t \to 0} \frac{\frac{t^3 + 0^2}{|t|+0}}{t}$$

$$= \lim_{t \to 0} \frac{t^3}{t|t|} = \lim_{t \to 0} \frac{t^2}{|t|} = \lim_{t \to 0} |t| = 0$$

de même $\dfrac{\partial g}{\partial y}(0,0) = 0$

$$\varphi(h,k) = \frac{g(h,k) - h\frac{\partial g}{\partial x}(0,0) - k\frac{\partial g}{\partial y}(0,0)}{\|(h,k)\|}$$

$$= \frac{\frac{h^3+k^3}{|h|+|k|}}{\sqrt{h^2+k^2}} = \frac{h^3+k^3}{(|h|+|k|)\sqrt{h^2+k^2}}$$

par passage en polaire $h = r\cos\theta$, $k = r\sin\theta$

$$\frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2(|\cos\theta|+|\sin\theta|)} = \frac{r(\cos^3\theta+\sin^3\theta)}{|\cos\theta|+|\sin\theta|} \xrightarrow[r\to 0]{} 0$$

donc g est différentiable en $(0,0)$

c) Non on ne sait pas sans calcul car comme g est différentiable, g peut être de classe $\mathcal{C}^1$ ou pas de classe $\mathcal{C}^1$.

Exo 4:

1) $T(x,y) = x^2y$

$$T(a+0.1, b+0.1) - T(a,b) \simeq dT_{(a,b)}(\Delta x, \Delta y)$$

$\Delta x = 0.1$
$\Delta y = 0.1$

$$\simeq dT_{(a,b)}(0.1, 0.1)$$

$$\simeq \frac{\partial f}{\partial x}(a,b) \times 0.1 + \frac{\partial f}{\partial y}(a,b) \cdot 0.1$$

$$\simeq 2ab \times 0.1 + a^2 \cdot 0.1.$$

2) Ici Δx et Δy sont infinitésimaux,

$$T(x_0+h, y_0+k) = T(x_0,y_0) + dT_{(x_0,y_0)}(h,k) + \|(h,k)\|\,\varphi(h,k)$$

avec $\varphi(h,k) \to 0$ $(h,k)\to(0,0)$

ici $\|(h,k)\| = (\Delta x, \Delta y) \to (0,0)$

$$\Delta T \simeq dT_{(x_0,y_0)}(h,k)$$

Exo 5 :

$z = 2xy + y^2 = f(x,y)$ $\quad \frac{\partial f}{\partial x} = 2y \quad \frac{\partial f}{\partial y} = 2x + 2y$

$$(T): \quad z = f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1)$$

$$(T): \quad z = 3 + 2(x-1) + 4(y-1)$$

$$(T): \quad \boxed{z = 2x + 4y - 3} \iff \boxed{2x + 4y - z = 3}$$

$$(P'): \quad 2x + 5y + 2z = 2$$

$\vec{n}_T \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ et $\vec{n}_{(P)} \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$

les 2 vecteurs normaux ne sont pas colinéaires

donc (T) n'est pas $\parallel$ à (P')

:

Exo 7 :

$$\sin^4 x = \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^4$$

$$= \frac{1}{16} \left(e^{ix} - e^{-ix} \right)^4$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$= \frac{1}{16} \left((e^{ix})^4 + 4 (e^{ix})^3 (-e^{-ix})^1 + 6 (e^{ix})^2 (-e^{ix})^2 + 4 (e^{ix}) (-e^{ix})^3 + (-e^{-ix})^4 \right)$$

$$= \frac{1}{16} \left(e^{4ix} - 4 e^{3ix} e^{-ix} + 6 e^{2ix} e^{-2ix} - 4 e^{ix} e^{-3ix} + e^{-4ix} \right)$$

$$= \frac{1}{16} \left(2\cos(4x) - 4 \cdot 2\cos(2x) + 6 \right)$$

$$\boxed{\sin^4 x = \frac{\cos(4x)}{8} - \frac{\cos(2x)}{2} + \frac{3}{8}}$$

$$\int \sin^4 x \, dx = \frac{\sin(4x)}{8 \times 4} - \frac{\sin(2x)}{2 \times 2} + \frac{3}{8} x + C \quad C \in \mathbb{R}$$

Exo 6 : L'équat de la ligne de niveau est $y + x^2 = 3$

donc $f(x,y) = y + x^2 + cte$

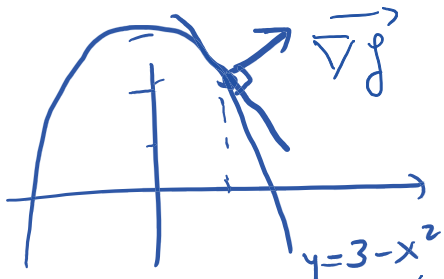
donc $\frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 1$

$\nabla f_{(1,2)} = \left(\frac{\partial f}{\partial x}(1,2), \frac{\partial f}{\partial y}(1,2) \right) = (2,1)$

$(2,1) \neq (1,1) \Rightarrow$ donc $\vec{u} \neq (1,1)$

Autre méthode



Tgte a pour coeff. dir. $\begin{pmatrix} 1 \\ -2x1 \end{pmatrix}$

ou tgte $\perp \vec{\nabla f}$ $= \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

et $(1,-2) \cdot (1,1) \neq 0$

$\Rightarrow \vec{u} \neq (1,1)$

Exo 8:

Préliminaire : $1+\tan^2\theta = 1+\left(\frac{\sin\theta}{\cos\theta}\right)^2 = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$\forall \quad \theta \in I$

1) $z = \dfrac{1}{1+i\tan\theta}$

2

$$z = \frac{1-i\tan\theta}{(1+i\tan\theta)(1-i\tan\theta)}$$

$$z = \frac{1-i\tan\theta}{1+\tan^2\theta} \quad \text{or} \quad 1+\tan^2\theta = \frac{1}{\cos^2\theta}$$

$$= \cos^2\theta\,(1-i\tan\theta)$$

$$\boxed{z = \cos^2\theta - i\cos\theta\sin\theta}$$

On lien $z = \cos\theta(\cos\theta - i\sin\theta)$

Si $\theta \in [0, \frac{\pi}{2}[\cup]\frac{3\pi}{2}; 2\pi[$

$|z| = \cos\theta$ et $\arg z = -\theta\,[2\pi]$

Sinon

$|z| = -\cos\theta$ et $\arg z = -\theta + \pi\,[2\pi]$

$$|z| = \sqrt{\cos^4\theta + \cos^2\theta\sin^2\theta}$$

$$= \sqrt{\cos^2\theta \times 1}$$

$$= |\cos\theta| = \begin{cases} \cos\theta & \text{si } \theta \in]0, \pi/2[\cup]\frac{3\pi}{2}, 2\pi[\\ -\cos\theta & \text{si } \theta \in]\frac{\pi}{2}, \frac{3\pi}{2}[\end{cases}$$

cas 1 : $|z| = \cos\theta$

$|z| = \cos\theta$

$z = \cos\theta(\cos\theta - i\sin\theta)$

$= \boxed{|z|\,e^{-i\theta}}$

$\Rightarrow \arg(z) = -\theta\,[2\pi]$

cas 2 :

$|z| = -\cos\theta$

$z = -\cos\theta(-\cos\theta + i\sin\theta)$

$z = -\cos\theta\, e^{i(\pi-\theta)}$

$\arg(z) = \pi - \theta\,[2\pi]$

$$\boxed{z = |z|\,e^{i(\pi-\theta)}}$$

B)

1)

$z_1 = 1 + i\tan\theta$

$|z_1| = \sqrt{1+\tan^2\theta} = \sqrt{\frac{1}{\cos^2\theta}} = \frac{1}{|\cos\theta|}$

Si $\cos\theta > 0$,

$|z_1| = \frac{1}{\cos\theta}$

$z_1 = \frac{1}{\cos\theta}(\cos\theta + i\sin\theta)$

$\arg(z_1) = \theta\ [2\pi]$

Si $\cos\theta < 0$

$|z_1| = -\frac{1}{\cos\theta}$

$z_1 = -\frac{1}{\cos\theta}(-\cos\theta - i\sin\theta)$

$z_1 = \frac{-1}{\cos\theta} e^{i(\pi+\theta)}$

$\arg(z_1) = \theta + \pi\ [2\pi]$

2) $z = 1/z_1$

$|z| = \frac{1}{|z_1|} = |\cos\theta|$

Si $\theta \in [0, \pi/2[$ $\Rightarrow$ $|z| = \cos\theta$

et $\arg(z) = -\arg(z_1)$

$= -\theta\ [2\pi]$