Exo 1 :

1) $z^2 + (-3-4i)z + 5i - 1 = 0$ (1)

$\Delta = (-3-4i)^2 - 4(5i-1)$

$= (3+4i)^2 - 20i + 4$

$= 9 + 24i - 16 - 20i + 4$

$= -3 + 4i$

$\delta^2 = (x+iy)^2 = (x^2-y^2) + 2ixy = -3+4i$

$\Leftrightarrow \begin{cases} x^2-y^2 = -3 \\ 2xy = +4 \\ x^2+y^2 = \sqrt{9+16} = 5 \end{cases} \Leftrightarrow \begin{cases} 2x^2 = 2 \\ 2y^2 = 8 \\ xy = 2 \end{cases} \Leftrightarrow \begin{cases} x = \pm 1 \\ y = \pm 2 \\ xy = 2 \end{cases}$

$\delta_1 = 1+2i$ ou $\delta_2 = -1-2i$

(1) $\Leftrightarrow$ $z = \dfrac{3+4i+1+2i}{2} = 2+3i$ ou $z = \dfrac{3+4i-1-2i}{2} = 1+i$

2) Cherchons 1 racine imaginaire pure pour $z^3 + (-3-2i)z^2 + (7-i)z - 10 - 2i = 0$ (E)

$z = ix \quad x \in \mathbb{R}$

(E) $\Leftrightarrow (ix)^3 + (-3-2i)(ix)^2 + (7-i)(ix) - 10 - 2i = 0$

(E) $\Leftrightarrow -ix^3 + (3+2i)x^2 + 7ix + x - 10 - 2i = 0$

(E) $\Leftrightarrow -ix^3 + 3x^2 + 2ix^2 + 7ix + x - 10 - 2i = 0$

(E) $\Leftrightarrow (3x^2+x-10) + i(-x^3+2x^2+7x-2) = 0$

(E) $\Leftrightarrow \begin{cases} 3x^2+x-10 = 0 \\ -x^3+2x^2+7x-2 = 0 \end{cases}$

$\Delta = 1^2 - 4\times 3\times(-10) = 121$

$x_1 = \dfrac{-1-11}{6} = -2$

$x_3 = \dfrac{-1+11}{6} = \dfrac{10}{6} = 5/3$

or $-(-2)^3 + 2\times 4 - 7\times 2 - 2 = 0$

(E) $\Leftrightarrow x = -2$

$-2i$

3)

Justifions que : $(z+2i)(z^2+(-3-4i)z+5i-1)$

$= z^3+(-3-2i)z^2+(-1-i+7)z$

développement ou système ou division $-10-2i$

$(z+2i)(z-(2+3i))(z-(1+i))=0$

$\Leftrightarrow \boxed{z=-2i \text{ ou } z=2+3i \text{ ou } z=1+i}$

Exo 2 :

$\left(\frac{z}{z+1}\right)^3 = 1 \quad (1)$

Posons $Z = \frac{z}{z+1}$

ou $z^3 = z^3+3z^2+3z+1$

$\Leftrightarrow 3z^2+3z+1=0$

$\Delta = 9-12 = -3$

$z = \frac{-3 \mp i\sqrt{3}}{6}$

$(1) \Leftrightarrow Z^3 = 1$

$(1) \Leftrightarrow \begin{cases} |Z|^3 = 1 \\ 3\arg(Z) = 0 + 2k\pi \end{cases}$

$(1) \Leftrightarrow Z=1 \text{ ou } Z = -\frac{1}{2}+i\frac{\sqrt{3}}{2} \text{ ou } Z = -\frac{1}{2}-i\sqrt{3}/2$

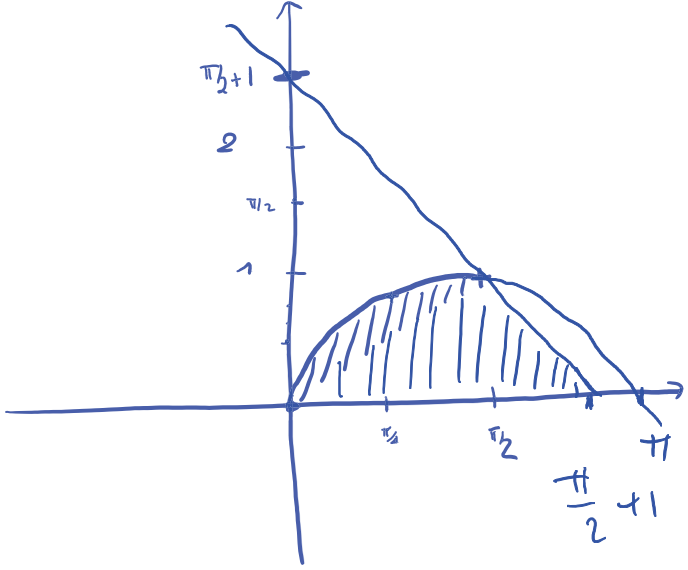
$(1) \Leftrightarrow \frac{z}{z+1}=1 \text{ ou } \frac{z}{z+1} = -\frac{1}{2}+i\frac{\sqrt{3}}{2} \text{ ou } \frac{z}{z+1} = -\frac{1}{2}-i\frac{\sqrt{3}}{2}$

$(1) \Leftrightarrow z = z+1 \text{ ou } z = \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)z + \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \text{ ou } z = \left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)(z+1)$

$(1) \Leftrightarrow \text{impossible ou } \left(\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)z = -\frac{1}{2}+i\frac{\sqrt{3}}{2} \text{ ou } \left(\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)z = -\frac{1}{2}-i\frac{\sqrt{3}}{2}$

$(1) \Leftrightarrow z = \frac{-1/2+i\sqrt{3}/2}{3/2-i\sqrt{3}/2} \text{ ou } z = \frac{-1/2-i\sqrt{3}/2}{3/2+i\sqrt{3}/2}$

Exo3:



Δ

Δ: $y = -x + \pi/2 + 1$

$$\iint_D y\,dx\,dy = \int_{x=0}^{\pi/2} \left(\int_{y=0}^{\sin x} y\,dy \right) dx + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+1} \left(\int_0^{-x+\frac{\pi}{2}+1} y\,dy \right) dx$$

$$= \int_{x=0}^{\pi/2} \left[\frac{y^2}{2} \right]_0^{\sin x} dx + \int_{\frac{\pi}{2}}^{\pi/2+1} \left[\frac{y^2}{2} \right]_0^{-x+\frac{\pi}{2}+1} dx$$

$$= \int_{x=0}^{\pi/2} \frac{\sin^2 x}{2}\,dx + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi/2+1} \left(-x+\frac{\pi}{2}+1\right)^2 dx$$

$$= \int_{x=0}^{\pi/2} \frac{1-\cos 2x}{4} + \frac{1}{2}\left[-\frac{1}{3}\left(-x+\frac{\pi}{2}+1\right)^3 \right]_{\pi/2}^{\pi/2+1}$$

$$= \left[\frac{1}{4}x - \frac{1}{4}\frac{\sin 2x}{2} \right]_0^{\pi/2} - \frac{1}{6}\left(-(1)^3\right)$$

$$= \frac{\pi}{8} - \frac{1}{8}\sin\pi + \frac{1}{6}$$

or $\cos^2 x + \sin^2 x = 1$

$\cos^2 x - \sin^2 x = \cos 2x$

$\frac{1-\cos(2x)}{2} = \sin^2 x$

Ex04:

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;/\; x \geq 0,\ y \geq 0 \text{ et } \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

$$= \left\{ (r,\theta) \;/\; 0 \leq \theta \leq \pi/2 \quad 0 \leq r \leq 1 \right\}$$

$$x = ar\cos\theta$$

$$y = br\sin\theta$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{bmatrix} = abr\cos^2\theta + abr\sin^2\theta$$

$$\Rightarrow \quad J = abr \quad \text{or } a \geq 0,\ b \geq 0,\ r \geq 0$$

$$|J| = abr$$

ainsi $$I = \iint_D 2x\,dx\,dy = \int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{1} 2ar\cos\theta\,(abr)\,dr \right) d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{1} 2a^2br^2\cos\theta\,dr \right) d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \cos\theta \left[2a^2b\frac{r^3}{3} \right]_0^1 d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \frac{2}{3}a^2b\cos\theta\,d\theta = \frac{2}{3}a^2b\left[\sin\theta\right]_{\theta=0}^{\theta=\pi/2}$$

$$I = 2/3\,a^2b$$

Exo 5:

- $f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$
  $(x,y) \longmapsto xy+6$

Surjection

$\forall a \in \mathbb{R}^2$,

$f(1, a-6) = a-6+6 = a$

donc f est surjective $\Rightarrow$ f non bijective

Injection

$f(-2,3) = 0$ donc $(-2,3) \neq (-3,2)$
$f(-3,2) = 0$

f non injective

- $f_2 : \mathbb{R} \longrightarrow \mathbb{R}^2$
  $x \longmapsto (x^2, -x)$

Surjection :

$(2,3)$ n'a pas d'antécédent par f :
En effet $(x^2, -x) = (2,3)$

$\Leftrightarrow \begin{cases} x = \pm\sqrt{2} \\ x = -3 \end{cases}$ impossible

f non surjective

Injection

Soient $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}$ /

$f(x_1) = f(x_2)$

$\Leftrightarrow (x_1^2, -x_1) = (x_2^2, -x_2)$

$\Leftrightarrow \begin{cases} x_1^2 = x_2^2 \\ -x_1 = -x_2 \end{cases}$

$\Leftrightarrow x_1 = x_2$

f injective

f non bijective

- $f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
  $(x,y) \longmapsto (2x+y, x-2y)$

$\forall (a,b) \in \mathbb{R}^2$ $\begin{cases} 2x+y = a & (L_1) \\ x-2y = b & (L_2) \end{cases}$

$(L_1) - 2(L_2)$

$\Leftrightarrow \begin{cases} 5y = a-2b \\ 5x = 2a+b \end{cases}$ $2(L_1)+(L_2)$

$\Leftrightarrow \begin{cases} x = \frac{2a+b}{5} \\ y = \frac{a-2b}{5} \end{cases}$

$\exists ! (x,y) \in \mathbb{R}^2$ / $f(x,y) = (a,b)$

donc f est surjective et injective donc bijective

et $f^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(a,b) \longmapsto \left(\frac{2a+b}{5}, \frac{a-2b}{5}\right)$

Exo C:

$$f(x) = \frac{x}{x^2+1}$$

f dérivable sur $[0,+\infty[$

et $f'(x) = \dfrac{1.(x^2+1) - x.2x}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$

x	0	x_1	1	2	x_2	$+\infty$
f'		$+$	$\oplus$	$-$		
f	0	$\nearrow$	$1/2$	$\searrow$		0

$f(x) = 1$ n'a pas de solution réelle

⇔ $x = x^2+1$

⇔ $x^2 - x + 1 = 0$ $\Delta = (-1)^2 - 4 < 0$

. f n'est pas surjective car ou bien f n'est pas injective car ... donc f n'est pas bijective

ou $f(\mathbb{R}^+) =]0, \frac{1}{2}]$ d'après le tabl. de variation.

2) $f([0,2]) = \{ f(x),\ x \in [0,2] \} = [0, \frac{1}{2}]$ d'après tableau de variation car f continue.

4) $f^{-1}([0, \frac{1}{4}]) = \{ x \in [0,+\infty[\ /\ 0 \leq f(x) \leq 1/4 \}$

$= \{ \text{——} \ /\ 0 \leq \dfrac{x}{x^2+1} \leq \dfrac{1}{4}$

$\Leftrightarrow 0 \leq x \leq \frac{1}{4}(x^2+1)$

$\Leftrightarrow \begin{cases} x \geq 0 \\ 4x \leq x^2+1 \end{cases} \Leftrightarrow \begin{cases} x \geq 0 \\ x^2 - 4x + 1 \geq 0 \end{cases}$

$\Delta = (-4)^2 - 4 \times 1 = 16 - 4 = 12$

$x_1 = \dfrac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$

$x_2 = \dfrac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$

x	0	x_1	x_2	$+\infty$
	$+$	0	$-$	0

$\Leftrightarrow \begin{cases} x \geq 0 \\ x \geq x_2 \text{ ou } x \leq x_1 \end{cases}$

$f^{-1}([0, \frac{1}{4}]) = [0, x_1] \cup [x_2, +\infty[$

Exo 7

$$\left(\frac{2z}{z+1}\right)^3 = 1$$

On pose $Z = \frac{2z}{z+1}$

$Z^3 = 1 \Leftrightarrow Z = 1$ ou $Z = \frac{-1+i\sqrt{3}}{2} = e^{\frac{2i\pi}{3}}$ ou $Z = \frac{-1-i\sqrt{3}}{2} = e^{-\frac{2i\pi}{3}}$

$Z = \frac{2z}{z+1} \Leftrightarrow Z(z+1) = 2z$

$\Leftrightarrow z = \frac{-Z}{Z-2}$

$(E) \Leftrightarrow z = \frac{-1}{-1} = 1$

ou $z = \frac{-e^{2i\pi/3}}{e^{2i\pi/3} - 2}$ ou $z = \frac{-e^{-\frac{2i\pi}{3}}}{e^{-\frac{2i\pi}{3}} - 2}$