DS 1 M2.2 2024

Exercice 1 :

$P(x) = x^3 - 2x^2 - x + 2$

1) $P(1) = 1 - 2 - 1 + 2 = 0$ donc 1 est racine de P.

2)
$$\begin{array}{l|l} x^3 - 2x^2 - x + 2 & x - 1 \\ -(x^3 - x^2) & \hline \\ \hline -x^2 - x + 2 & x^2 - x - 2 \\ -(-x^2 + x) & \\ \hline -2x + 2 & \end{array}$$

donc $P = (x-1)(\underbrace{x^2 - x - 2}_{\Delta = 1 + 8 = 9})$

$x_1 = \frac{1+3}{2} = 2$

$x_2 = \frac{1-3}{2} = -1$

ainsi $P = (x-1)(x+1)(x-2)$

3) $R = \frac{6}{P(x)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{c}{x-2}$

$(x-1)R\big|_{x=1} = a = -3$

$(x+1)R\big|_{x=-1} = b = 1$

$(x-2)R\big|_{x=2} = c = 2$

donc $R = \frac{-3}{x-1} + \frac{1}{x+1} + \frac{2}{x-2}$

$$\sum_{k=3}^{n} R(k) = -3\sum_{k=3}^{n} \frac{1}{k-1} + \sum_{k=3}^{n} \frac{1}{k+1} + 2\sum_{k=3}^{n} \frac{1}{k-2}$$

$k-1 \leftarrow k+1$

$\Leftrightarrow k \leftarrow k+2$

$k-1 \leftarrow k-2$

$\Leftrightarrow k \leftarrow k-1$

$$= -3\sum_{k=3}^{n} \frac{1}{k-1} + \sum_{k=5}^{n+2} \frac{1}{k-1} + 2\sum_{k=2}^{n-1} \frac{1}{k-1}$$

$$= -3\left(\underbrace{\frac{1}{2} + \frac{1}{3}}_{5/6} + \sum_{5}^{n-1} \frac{1}{k-1} + \frac{1}{n-1}\right) + \left(\sum_{5}^{n-1} \frac{1}{k-1} + \frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1}\right)$$

$$+ 2\left(\sum_{5}^{n-1} \frac{1}{k-1} + \underbrace{1 + \frac{1}{2} + \frac{1}{3}}_{11/6}\right)$$

$$= \frac{7}{6} - 2\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1}.$$

Esercizio 2:

1) $R_1 = \dfrac{9x}{(x-2)^2(x+1)^2} = \dfrac{a_1}{x-2} + \dfrac{a_2}{(x-2)^2} + \dfrac{b_1}{x+1} + \dfrac{b_2}{(x+1)^2}$

$$(x-2)^2 R_1 \Big|_{x=2} = \frac{18}{9} = 2 = a_2$$

$$(x+1)^2 R_1 \Big|_{x=-1} = \frac{-9}{9} = -1 = b_2$$

$$\lim_{+\infty} x R_1 = 0 = a_1 + b_1$$

$$R_1(0) = 0 = -\frac{a_1}{2} + \frac{2}{4} + b_1 - 1$$

$$\Rightarrow \begin{cases} a_1 = -b_1 \\ 0 = \frac{3b_1}{2} - \frac{1}{2} \end{cases} \qquad b_1 = \frac{1}{3} \qquad a_1 = -\frac{1}{3}$$

2] $$R_2 \quad \frac{3x^2+5}{x^3(x^2+1)} = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{bx+c}{x^2+1}$$

$$(x^2+1) R_2 \Big|_{x=i} = bi + c = \frac{2}{i^3} = 2i$$

$$b = 2 \qquad c = 0.$$

$$x^3 R_2 \Big|_{x=0} = a_3 = 5$$

$$\lim_{+\infty} x R_2 = 0 = a_1 + b \qquad a_1 = -2$$

$$R_2(1) = \frac{8}{2} = 4 = -2 + a_2 + 5 + \frac{2}{2}$$

$$\underline{a_2 = 0}$$

3] $$\frac{x^4+2}{x^3+4x} = \frac{x^4+2}{x(x^2+4)} = E + \frac{a}{x} + \frac{bx+c}{x^2+4}.$$

E: $$\begin{array}{r|l} x^4+2 & x^3+4x \\ -(x^4+4x^2) & \hline & x \\ \hline -4x^2+2 & \end{array} \qquad E = x.$$

$$R_3 = \frac{-4x^2+2}{x(x^2+4)}$$

$$\text{Donc } \frac{X^4+2}{X^3+4X} = X + \frac{1/2}{X} + \frac{-5/2\,X}{X^2+4}$$

$$(x^2+4)R_3\Big|_{x=2i} = 2bi + c = \frac{10}{2i} = -5i$$

$$b = -5/2 \quad c = 0$$

$$xR_3\Big|_{x=0} = a = \frac{2}{4} = \frac{1}{2}$$

Dans $\mathbb{C}(X)$.

$$\frac{-4X^2+2}{X(X^2+4)} = \frac{a}{X} + \frac{\alpha}{X-2i} + \frac{\beta}{X+2i}$$

↳ $a = 1/2$ comme dans $\mathbb{R}$, puis en $2i$, $\alpha = \frac{18}{2i(4i)} = \frac{-9}{4}$ et en $-2i$, $\beta = \frac{18}{2i.4i} = \frac{-9}{4}$

4.) $f(x) = \frac{x^2(x-2)}{(x^2-4x+4)^2} = \frac{x^2(x-2)}{(x-2)^4} = \frac{X^2}{(x-2)^3}$

Or $X^2 = (X-2)^2 + 4X - 4 = (X-2)^2 + 4(X-2) + 4$

Donc $f(x) = \frac{(X-2)^2+4(X-2)+4}{(X-2)^3} = \frac{1}{(X-2)} + \frac{4}{(X-2)^2} + \frac{4}{(X-2)^3}$

Exercice 3 : $f : E \to E'$.

1] Soit $v \in \operatorname{Im} f$ sev de E' $\quad v = f(u)$

$$f(0_E) + v = f(0_E) + f(u) = f(u + 0_E)$$

$$= f(u) = v$$

donc $f(0_E) = 0_{\operatorname{Im} f} = 0_{E'}$

2) Soit $F = \mathrm{Im} f$.

$f(0_E) \in \mathrm{Im} f$ donc $\mathrm{Im} f$ non vide.

Soient $v_1, v_2 \in \mathrm{Im} f$ et $\alpha \in \mathbb{R}$.

$\exists u_1 \in E$ / $v_1 = f(u_1)$

$\exists u_2 \in E$ / $v_2 = f(u_2)$

$v_1 + \alpha v_2 = f(u_1) + \alpha f(u_2)$ ↙ linéarité de f.

$= f(u_1 + \alpha u_2)$

donc $v_1 + \alpha v_2 \in \mathrm{Im} f$.

35 min

Exercice 4 :

$$p : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2x - 2z \\ y \\ x - z \end{pmatrix} \end{cases}$$

1) $u, v \in \mathbb{R}^3$ $\quad u \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad v \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\alpha \in \mathbb{R}$

$$p(u) + \alpha p(v) = \begin{pmatrix} 2x - 2z \\ y \\ x - z \end{pmatrix} + \alpha \begin{pmatrix} 2a - 2c \\ b \\ a - c \end{pmatrix}$$

$$= \begin{pmatrix} 2x - 2z + \alpha(2a - 2c) \\ y + \alpha b \\ x - z + \alpha(a - c) \end{pmatrix}$$

(1)

$$u + \alpha v = \begin{vmatrix} x + \alpha a \\ y + \alpha b \\ z + \alpha c \end{vmatrix} \qquad p(u+\alpha v) = \begin{vmatrix} 2(x+\alpha a) - 2(z+\alpha c) \\ y + \alpha b \\ x + \alpha a - (z + \alpha c) \end{vmatrix} \quad (2)$$

(1) = (2) donc p est linéaire.

2) $\forall u \in \mathbb{R}^3$

$$p \circ p(u) = p(p(u)) = p \begin{vmatrix} 2x - 2z = X \\ y = Y \\ x - z = Z \end{vmatrix}$$

$$= \begin{vmatrix} 2X - 2Z = 4x - 4z - 2x + 2z = 2x - 2z = p(u) \\ Y = y \\ X - Z = 2x - 2z - (x - z) = x - z \end{vmatrix}$$

donc $p \circ p = p$

3) $\underline{\text{Ker } p}$: $u \in \mathbb{R}^3$ / $p(u) = 0$

$$\begin{cases} 2x - 2z = 0 \\ y = 0 \\ x - z = 0 \end{cases} \iff \begin{cases} y = 0 \\ x - z = 0 \end{cases}$$

donnée de $\mathbb{R}^3$: $\text{Ker } p = \text{Vect} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

4) $\text{rg } p = \dim E - \dim \text{Ker } p = 2$

5) . $\dim \text{Ker } p + \dim \underline{\text{Im } p} = \dim \mathbb{R}^3$.

- Soit $u \in \ker p \cap \operatorname{Im} p$.

  $u \in \ker p$ donc $p(u) = 0$

  $u \in \operatorname{Im} p$ donc $\exists v \in \mathbb{R}^3 / \quad u = p(v)$

$$0 = p(u) = p \circ p(v) = p(v) = u$$

$$\ker p \cap \operatorname{Im} p = \{0\}$$

$$\boxed{\ker p \oplus \operatorname{Im} p = \mathbb{R}^3}$$

Somme

Exercice 5 : $f : E \to E'$

1) f injective $\Leftrightarrow \dim \ker f = 0$ donc $\dim \operatorname{Im} f = \dim E$

ou $\dim \operatorname{Im} f \leq \dim E'$.

Vrai

4) Non Th du rang $\operatorname{rg} f \leq \dim E$.

3) $\ker f = \{ f \in \mathcal{C}^2 / 2f'' = 0 \}$

$$f'' = 0 \rightsquigarrow f' = k \longrightarrow f = kx + c$$

Faux

4) f non linéaire : FAUX

$f(2,2,2) = (4,2,2)$

et $2 \cdot f(1,1,1) = 2(1,1,1) \neq (4,2,2)$.