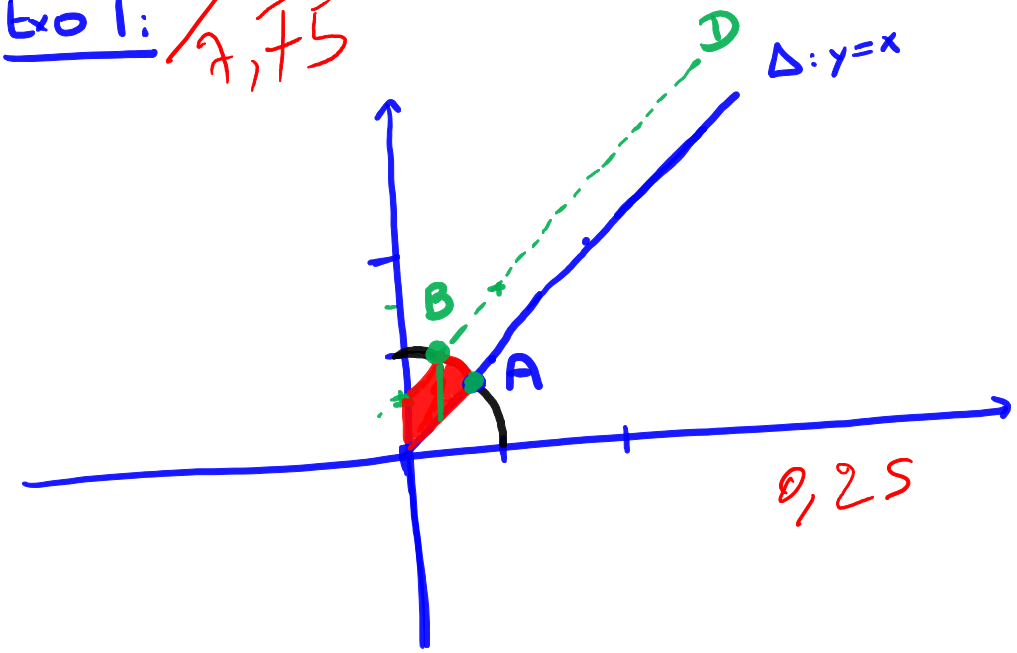
Exo 1: /7,75



D: $y = x + 1/2$

[redacted] = D1

Déterminons les coordonnées des points d'intersection de Δ et C puis de D et C

1,75

- dans le 1/4 de plan supérieur l'intersection de Δ et C est $A\begin{pmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}$ 0,25

---

de D et C :

$$\begin{cases} 1 = x^2+y^2 \\ y = x+1/2 \end{cases} \Leftrightarrow \begin{cases} 1 = x^2+x^2+x+1/4 \\ y = x+1/2 \end{cases} \Leftrightarrow \begin{cases} 2x^2+x-3/4=0 \\ y = x+1/2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \frac{-1+\sqrt{7}}{2} \text{ car } x \geq 0 \\ y = x+1/2 \end{cases}$$

$\Delta = 1^2 - 4\times 2\times(-3/4)$
$= 1+6 = 7$

$B\left(\frac{-1+\sqrt{7}}{4}\right)$ $\sqrt{7}/2$ 0,5

0,25 pour somme de 2 $\int$

$$I = \int_{x=0}^{x_B} \left(\int_{y=x}^{x+1/2} f(x,y)\,dy \right) dx + \int_{x=x_B}^{x=x_A} \left(\int_{y=x}^{+\sqrt{1-x^2}} f(x,y)\,dy \right) dx$$

0,25 0,25

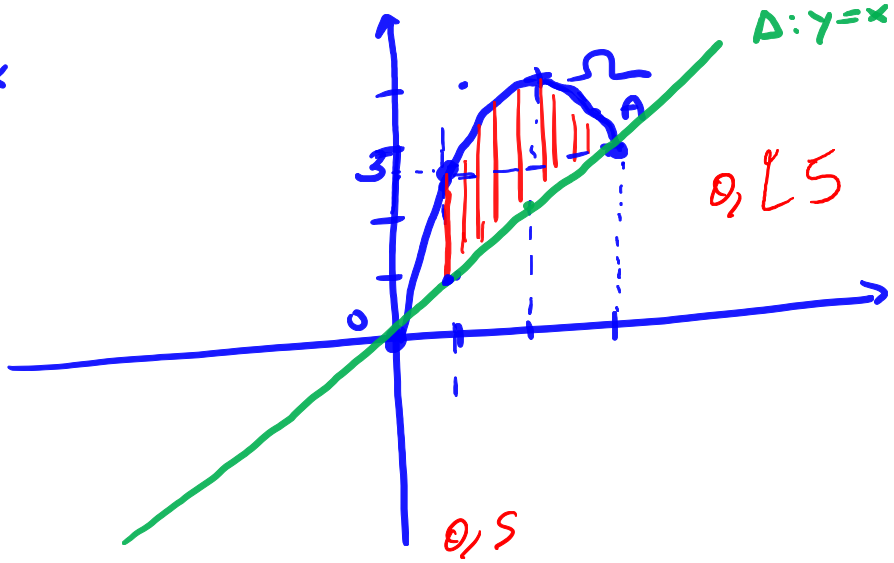
D_2

(P) $y = -x^2+4x$

$\Leftrightarrow y = -(x-2)^2+4$

$\Leftrightarrow y-4 = -(x-2)^2$ d'où parabole de sommet $\Omega\begin{pmatrix}2\\4\end{pmatrix}$

(Δ): $y=x$



0,25

$\Delta \cap P = \{O, A\}$

$O\begin{pmatrix}0\\0\end{pmatrix}$ $A\begin{pmatrix}3\\3\end{pmatrix}$

$D_2 =$ [redacted]

$$\iint_{D_2} x \, dx\,dy = \int_{x=1}^{3} \left(\int_{y=x}^{-x^2+4x} x \, dy \right) dx$$

$$= \int_{x=1}^{3} \left[xy \right]_{y=x}^{y=-x^2+4x} dx$$

0,25

$$= \int_{x=1}^{3} x(-x^2+4x) - x^2 \, dx$$

0,25

$$= \int_{x=1}^{3} -x^3+3x^2 \, dx$$

$$= \left[\frac{-x^4}{4} + \frac{3x^3}{3} \right]_1^3$$

0,25

$$= -\frac{3^4}{4} + 3^3 - \left(\frac{-1}{4}+1\right)$$

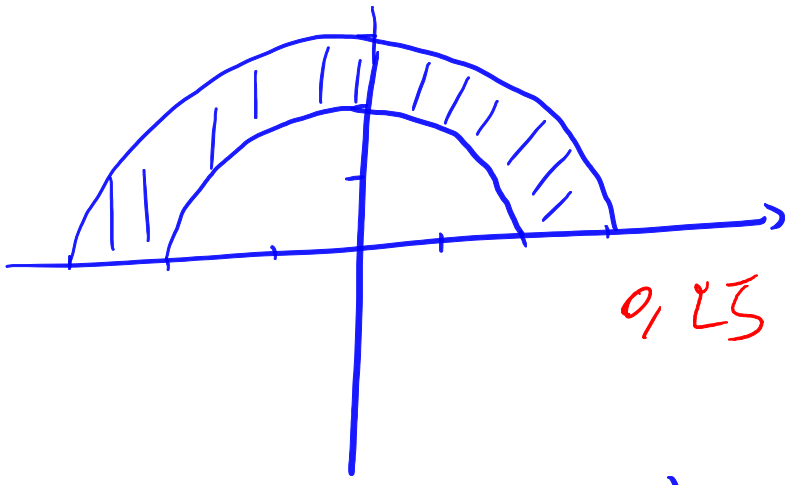
$$= -3^4/4 + 27 - 3/4 \; (= 6)$$

0,25

1,75

Non exigé

D_3



0,25

$$D_3 = \{(x,y) \in \mathbb{R}^3 / \; y \geq 0 \text{ et } 4 \leq x^2+y^2 \leq 9\}$$

$$\iint_{D_3} xy^2 \, dx\,dy = \int_{\theta=0}^{\pi} \left(\int_{r=2}^{3} (r\cos\theta)(r\sin\theta)^2 \, r \, dr \right) d\theta$$

0,25 / 0,5

$$= \int_{\theta=0}^{\pi} \left(\int_{r=2}^{3} r^4 \cos\theta \sin^2\theta \, dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi} \cos\theta \sin^2\theta \left[\frac{r^5}{5} \right]_{r=2}^{r=3} d\theta \quad /0,5$$

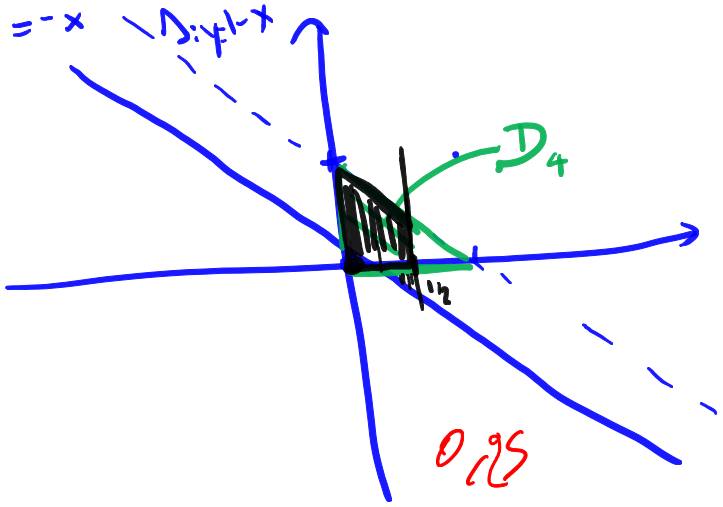
$$= \left(\frac{3^5 - 2^5}{5} \right) \int_{\theta=0}^{\pi} \underbrace{\cos\theta \sin^2\theta}_{u' u^2} \, d\theta$$

$$= \frac{3^5 - 2^5}{5} \cdot \left[\frac{\sin^3\theta}{3} \right]_{\theta=0}^{\theta=\pi} = 0 \quad /0,25$$

0,5

1,5

3/4



Δ : y = 1-x

$-x \leq y \leq 1-x$

$0 \leq x \leq 1$

$-x \leq y \leq 1-x$

0,25

$\begin{cases} u = x \\ y = (1-u)v = v - uv \end{cases} \Rightarrow v = \dfrac{y}{1-u} = \dfrac{y}{(1-x)}$

ici $x = u$

$1 - x \geq v$

$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -v & 1-u \end{vmatrix} = 1-u$ 0,5

$|J| = 1-u$ car $u \leq 1$ 0,25

$\Rightarrow dx\,dy = (1-u)\,du\,dv$

$0 \leq x + y \leq 1 \iff 0 \leq u + (1-u)v \leq 1$

$\iff -\dfrac{u}{1-u} \leq v \leq 1$

$\iint = \iint_{\mathcal{D}} x \dfrac{y}{(1-x)} \times \dfrac{1}{1-x}\,du\,dy$ 0,5

$= \iint_{\mathcal{D}'} u\,v\,du\,dv = \int_0^{1/2} \left(\int_{-\frac{u}{1-u}}^{1} uv\,dv \right) du$ 0,5

0,25

2,75

$= \int_0^{1/2} \dfrac{u}{2} \left(1 - \dfrac{u^2}{(1-u)^2} \right) du$ 0,5

Non exigé.

$$= \frac{1}{2}\int_0^{1/2} u\,\frac{1-2u}{(1-u)^2}\,du$$

$$= \frac{1}{2}\int_0^{1/2} \frac{2u^2+u}{(1-u)^2}\,du = \frac{1}{2}\int_0^{1/2} \frac{-2(1-u)^2+2-3u}{(1-u)^2}\,du$$

$$= \frac{1}{2}\int_0^{1/2} -2+\frac{3(1-u)-1}{(1-u)^2}\,du = \frac{1}{2}\left[-2u-3\ln|1-u|-\frac{1}{1-u}\right]_0^{1/2}$$

$$= \frac{1}{2}\left(-1-3\ln\frac{1}{2}-2+1\right) = -1+\frac{3}{2}\ln 2$$

Exo 2 — 1,75

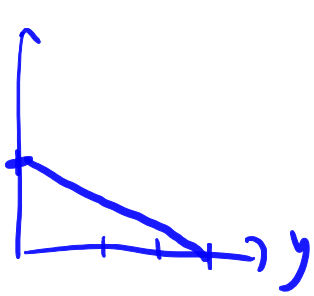
$$I = \iiint_S f(x,y,z)\,dx\,dy\,dz$$



$$= \int_{x=0}^{2}\left(\int_{y=0}^{3}\left(\int_{z=0}^{1-\frac{1}{3}y} f(x,y,z)\,dz\right)dy\right)dx$$

0,25 0,25 0,25

x fixé, coupe

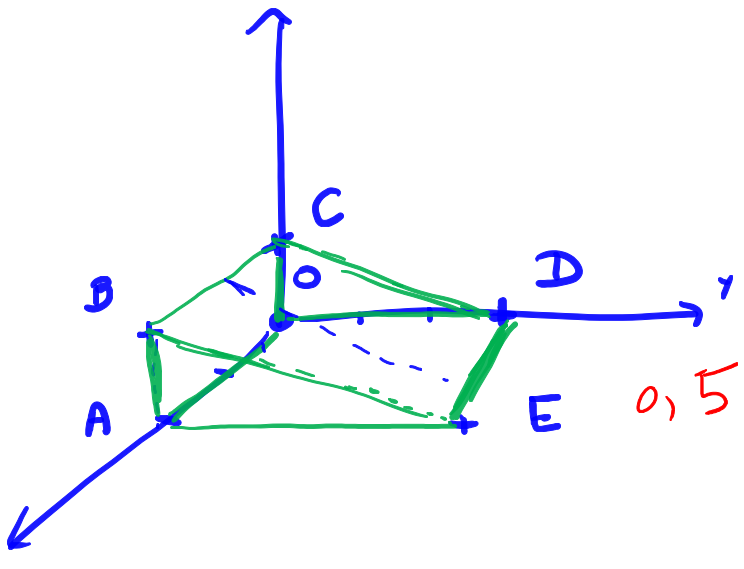


+ explication 0,5

0

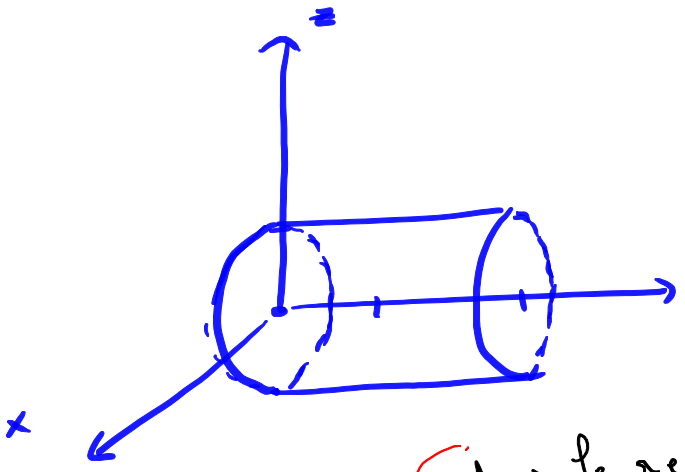
$$z = \frac{0-1}{3-0}y+1$$

$$z = -\frac{1}{3}y+1$$



0,5

Exo 3: /2



0,25

0,5

dans le repère 

les coordonnées cylindriques sont $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ et $dS = r\,dr\,d\theta\,dz$

donc ici on va prendre $\begin{cases} x = r\cos\theta \\ z = r\sin\theta \end{cases}$ et $dS = r\,dr\,dy\,d\theta$

$$I = \iiint_S xy\,dx\,dy\,dz$$

$$= \int_{y=0}^{2} S_y\,dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(\int_{y=0}^{2} r\cos\theta \cdot y\, r\,dy \right) dr \Big) d\theta$$

(0,25 ; 0,25 ; 0,25 ; 0,25)

$$= \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{1} r^2\cos\theta \left[\frac{y^2}{2}\right]_0^2 dr \right) d\theta$$

$$= \int_0^{2\pi} \cos\theta \, d\theta \times \int_0^1 r^2 \, dr \times 2 = \left[\sin\theta\right]_0^{2\pi} \times \left[\frac{r^3}{3}\right]_0^1 \times 2$$

$$= 0 \qquad 0,25$$

Exo 4 — 1,5

$I_1 \left(x = \sqrt{3}/2 \, , \; y = 1/2, \; z = 1 \right)$

. $r = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{3/4 + 1/4 + 1}$ 0,5

$\boxed{r = \sqrt{2}}$

. $\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$

$z = r\cos\theta$

$\Rightarrow \cos\theta = 1/\sqrt{2} = \sqrt{2}/2$

$\theta \in [0, \pi] \Rightarrow \boxed{\theta = \pi/4}$ 0,5

$x = r \sin\theta \cos\varphi$

$\Rightarrow \sqrt{3}/2 = \sqrt{2} \cdot \frac{\sqrt{2}}{2} \cos\varphi$

ET

$y = r \sin\theta \sin\varphi$

$1/2 = \frac{\sqrt{2}}{2} \cdot \sqrt{2} \sin\varphi$

$\Rightarrow \begin{cases} \cos\varphi = \sqrt{3}/2 \\ \sin\varphi = 1/2 \end{cases}$

$\Rightarrow \boxed{\varphi = \pi/6}$ 0,5

Exo 5 /1,5

1) . $\mathbb{R}^2$ est simplement connexe

• $P(x,y) = (1+x+y)e^x$
  $Q(x,y) = e^x$) 0,25

$\Rightarrow \omega$ est exacte

0,25

$\frac{\partial P}{\partial y} = e^x$

$\frac{\partial Q}{\partial x} = e^x$) 0,25

0,75

2) cherchons des primitives de ω

$\begin{cases} \partial u/\partial x = (1+x+y)e^x \quad (1) \\ \partial u/\partial y = e^x \quad (2) \end{cases}$) 0,25

1,75

2) $\Rightarrow u(x,y) = ye^x + c(x)$

En reinjectant dans (1)

$\Rightarrow \cancel{ye^x} + c'(x) = (1+x)e^x + \cancel{xe^x}$

$\Rightarrow c'(x) = (1+x)e^x$) via IPP

$\Rightarrow c(x) = xe^x + a,$

on bien (+Pmg)

1) $\Rightarrow u = e^x + \int xe^x dx + ye^x + A(y)$

Par exemple on choisit

$u(x,y) = ye^x + xe^x$ | 0,75 avec justification, sinon 0

Ainsi

$\oint_{(\Gamma)} \omega = u(B) - u(A)$ car ω est exacte 0,5

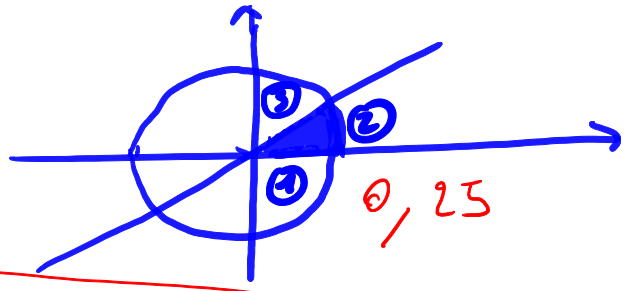
$= u(1,1) - u(0,0)$

$B\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$ $A\left(\begin{pmatrix}0\\0\end{pmatrix}\right)$

$\boxed{\oint_{(\Gamma)} \omega = 2e}$ 0,25

Exo 6 : 4, 5

1) Par définition :



0,25

Paramétrisons le bord de (D) ci-contre :

② $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$ $0 \le t \le \pi/4$ 0,25 $\quad \begin{cases} dx = -\sin t\, dt \\ dy = \cos t\, dt \end{cases}$ / 0,25

$$\int_{②} \omega = \int_{t=0}^{\pi/4} (\cos t + \sin t)(-\sin t) - \sin t (\cos t)\, dt \quad 0,25$$

$$= \int_{t=0}^{\pi/4} -2 \cos t \sin t - \sin^2 t \, dt$$

$$= -\int_{t=0}^{\pi/4} \sin(2t) + \sin^2 t \, dt$$

$\cos^2 - \sin^2 = \cos(2t)$
$\cos^2 + \sin^2 = 1$
$\sin^2 t = \frac{1-\cos(2t)}{2}$

$$= -\int_{t=0}^{\pi/4} \sin(2t) + \frac{1-\cos(2t)}{2} dt$$

$$= -\left[\underbrace{-\frac{\cos(2t)}{2}}_{\text{ou } \sin^2 t \text{ 0,25}} + \underbrace{\frac{1}{2}t}_{0,25} - \frac{1}{2}\frac{\sin(2t)}{2} \right]_{t=0}^{\pi/4}$$

$$= -\left[0 + \frac{\pi}{8} - \frac{1}{4} \times 1 - \left(-\frac{1}{2} + 0 \right) \right]$$

$$= -\left[\pi/8 - \frac{1}{4} + \frac{1}{2} \right]$$

$$= -\pi/8 - \frac{1}{4} \,/\, 0,25$$

(1)

① $\begin{cases} x = t \\ y = 0 \end{cases}$ $t \in [0;1]$ 0,25 $\begin{cases} dx = dt \\ dy = 0 \end{cases}$ 0,25

$\int_{①} \omega = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = 1/2$ 0,25

③ $\begin{cases} x = t \\ y = t \end{cases}$ 0,25 $t \in [0; \frac{\sqrt{2}}{2}]$ 0,25 $\begin{cases} dx = dt \\ dy = dt \end{cases}$

$\int_{①} \omega = \int_0^{\frac{\sqrt{2}}{2}} 2t - t\,dt = \int_{\frac{\sqrt{2}}{2}}^0 t\,dt = \left[\frac{t^2}{2}\right]_{\frac{\sqrt{2}}{2}}^0 = -1/4$ 0,25

$\int_{\partial D} \omega = \int_{①} + \int_{②} + \int_{③} = -\frac{\pi}{8} - \frac{1}{4} + \frac{1}{2} - \frac{1}{4} = -\frac{\pi}{8}$ 0,25

2) Par la formule de Green Riemann

$\int_{\partial D} \omega = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ 0,25

$P(x,y) = x+y$, $\frac{\partial P}{\partial y} = 1$

$Q(x,y) = -y$, $\frac{\partial Q}{\partial x} = 0$ 0,25

$= \iint_D (0 \quad -1)\,dx\,dy$

$= \iint_D (-1)\,dx\,dy$

par passage en coordonnées polaires

① $= \int_{\theta=0}^{\pi/4} \left(\int_{r=0}^{1} (-1)\, r\,dr \right) d\theta$ 0,25

$= \int_{\theta=0}^{\pi/4} \left[\frac{-r^2}{2}\right]_{r=0}^{r=1} d\theta$

$= -1/2 \cdot \pi/4$

$= -\pi/8$ 0,25