Exo 1 : (2)

1) $\frac{4+i}{(2+2i)^2} = \frac{4+i}{4+8i-4}$

$= \frac{4+i}{8i}$

$= \frac{-4i+1}{+8}$

$= \frac{1}{8} - \frac{4i}{8}$

$= \frac{1}{8} - i/2$ } 0,75

$\frac{R+jL}{R+jL\omega} = \frac{(R+jL)(R-jL\omega)}{(R+jL\omega)(R-jL\omega)}$

$= \frac{R^2+L^2\omega + j(LR-L\omega R)}{R^2+L^2\omega^2}$

$= \frac{R^2+L^2\omega + jLR(1-\omega)}{R^2+L^2\omega^2}$ } 0,75

2)

$\left|\frac{RC\omega}{R+jL}\right| = \frac{|RC\omega|}{\sqrt{R^2+L^2}} = \frac{RC\omega}{\sqrt{R^2+L^2}}$ } 0,5

car $R, \omega, C > 0$

Exo 2 : (2,5)

1) $\frac{1}{z-1} = 1 + \frac{2i}{1-i} = \frac{1-i+2i}{1-i}$

$\Leftrightarrow z \neq 1$ et $\frac{1}{z-1} = \frac{1+i}{1-i}$

$\Leftrightarrow$ ____ et $z-1 = \frac{1-i}{1+i}$

$\Leftrightarrow$ ____ et $z-1 = \frac{(1-i)^2}{(1+i)(1-i)}$

$\Leftrightarrow$ ____ et $z-1 = \frac{1-2i-1}{2}$

$\Leftrightarrow$ ____ et $z = 1-i$ } 1,25

$S = \{1-i\}$

2) $1/z = 1/R + i\,1/C = \frac{C+iR}{RC}$

$\Leftrightarrow z \neq 0$ et $z = \frac{RC}{C+iR}$

$\Leftrightarrow$ ____ $z = \frac{RC(C-iR)}{C^2+R^2}$ 0,75

$S = \left\{\frac{RC^2 - iR^2C}{C^2+R^2}\right\}$

3) $z^2 + 3z + 4 = 0$ (3)

$\Delta = 9 - 4\times4 = 9 - 16 = -7$

$\Delta < 0$ donc (3) admet 2 racines complexes conjuguées :

$\frac{-3-i\sqrt{7}}{2}$ et $\frac{-3+i\sqrt{7}}{2}$

$S_3 = \left\{\frac{-3-i\sqrt{7}}{2}, \frac{-3+i\sqrt{7}}{2}\right\}$ } 0,5

Exo 3 : 2,75

1) $\dfrac{i}{-1+\sqrt{3}i} = \dfrac{i}{2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)} = \dfrac{1}{2}\dfrac{e^{i\pi/2}}{e^{2i\pi/3}}$ car $-\frac{1}{2}+i\frac{\sqrt{3}}{2} = e^{2i\pi/3}$

1,75

$= \frac{1}{2}\, e^{i\left(\pi/2 - \frac{2\pi}{3}\right)} = \frac{1}{2}\, e^{i\frac{3\pi-4\pi}{6}}$

$= \frac{1}{2}\, e^{-i\pi/6}$ → 0,75

$z^3 = \frac{1}{2}\, e^{-i\pi/6}$ (E)

(E) $\Leftrightarrow (\rho e^{i\theta})^3 = \frac{1}{2}\, e^{-i\pi/6}$

(E) $\Leftrightarrow \rho^3 e^{3i\theta} = \frac{1}{2}\, e^{-i\pi/6}$

(E) $\Leftrightarrow \begin{cases} \rho^3 = \frac{1}{2} \\ 3\theta = -\pi/6 + 2k\pi \end{cases}$ 0,5 $\Leftrightarrow \begin{cases} \rho = \left(\frac{1}{2}\right)^{1/3} = \frac{1}{2^{1/3}} \\ \theta = -\frac{\pi}{18} + \frac{2k\pi}{3} \end{cases}$

$S = \left\{ \underbrace{\frac{1}{2^{1/3}} e^{-i\pi/18}}_{(k=0)}, \underbrace{\frac{1}{2^{1/3}} e^{\frac{11i\pi}{18}}}_{(k=1)}, \underbrace{\frac{1}{2^{1/3}} e^{\frac{23i\pi}{18}}}_{(k=2)} \right\}$ 0,5

2) $|-3+i\sqrt{3}| = \sqrt{9+(\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

$-3+i\sqrt{3} = 2\sqrt{3}\left(\frac{-\sqrt{3}}{2}+\frac{i}{2}\right) = 2\sqrt{3}\, e^{5i\pi/6}$ 0,5

1 ainsi $(-3+i\sqrt{3})^{10} = 2^{10} 3^5 \left(e^{5i\pi/6}\right)^{10}$

$= 2^{10} 3^5 e^{\frac{50i\pi}{6}}$ 0,25

or $50 = 6\times 8 + 2$

$50\pi = 6\times 8\pi + 2\pi$

$\frac{50\pi}{6} = 8\pi + \pi/3$

$e^{\frac{50i\pi}{6}} = 1\times e^{i\pi/3}$

$\Rightarrow (-3+i\sqrt{3})^{10} = 2^{10} 3^5 e^{i\pi/3}$ 0,25

Exo 4 : (2)

$$I = \int_{y=0}^{1} \left(\int_{x=0}^{y} \left(\int_{z=y}^{1} x \, dz \right) dx \right) dy$$ 0,75

$$= \int_{y=0}^{1} \left(\int_{x=0}^{y} \left[x z \right]_{z=y}^{z=1} dx \right) dy$$

$$= \int_{y=0}^{1} \left(\int_{x=0}^{y} (x - xy) \, dx \right) dy$$

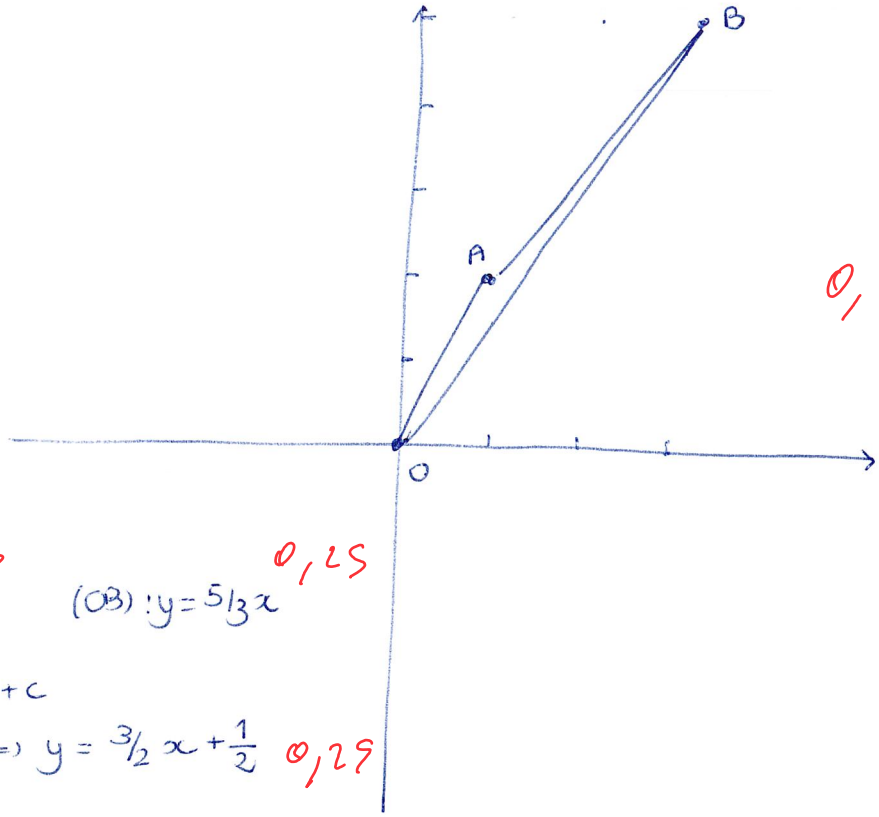
$$= \int_{y=0}^{1} \left[\frac{x^2}{2} - \frac{x^2}{2} y \right]_{x=0}^{x=y} dy$$

0,25 par étapes (0,75)

$$= \int_{y=0}^{1} \left[y^2/2 - y^3/2 \right] dy = \left[\frac{y^3}{6} - \frac{1}{2} \frac{y^4}{4} \right]_0^1 = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

0,25 0,25

Exo 6 : 1,75



0,25

(OA) : $y = 2x$ 0,25 (OB) : $y = 5/3\,x$ 0,25

(AB) : $y = \frac{5-2}{3-1}x + c$

$\Leftrightarrow y = 3/2\,x + \frac{1}{2}$ 0,25

$A\begin{pmatrix}1\\2\end{pmatrix}$ $B\begin{pmatrix}3\\5\end{pmatrix}$

$$\iint_D f(x,y)\,dx\,dy = \int_{x=0}^{1}\left(\int_{y=5/3\,x}^{2x} f(x,y)\,dy\right)dx + \int_{x=1}^{3}\left(\int_{y=5/3\,x}^{3/2\,x+1/2} f(x,y)\,dy\right)dx$$

0,75

Exo 7 : (3)

$u = xy$

$v = x + 1$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix}$$

$\Rightarrow J = |y \times 0 - x \times 1| = |-x| = |x| = x$ car x positif } 0,5

$$I = \iint_D \frac{xy}{x+1} \cdot x \, dx \, dy = \iint_R \frac{u}{v} \, du \, dv$$ } 0,5

avec $R = \{(u,v) \in \mathbb{R}^2 / \ 2 \leq v \leq 3 \text{ et } 1 \leq u \leq v\}$ } 0,5

$$I = \int_{v=2}^{3} \left(\int_{u=1}^{u=v} \frac{u}{v} \, du \right) dv$$ } 0,5

$$I = \int_{v=2}^{3} \left[\frac{u^2}{2v} \right]_{u=1}^{u=v} dv$$

$$= \int_{v=2}^{3} \left(\frac{v^2}{2v} - \frac{1}{2v} \right) dv = \int_{v=2}^{3} \left(\frac{v}{2} - \frac{1}{2} \cdot \frac{1}{v} \right) dv$$ } 0,5

$$= \left[\frac{v^2}{4} - \frac{1}{2} \ln|v| \right]_{v=2}^{v=3} = \frac{9}{4} - \frac{1}{2} \ln 3 - \left(\frac{4}{4} - \frac{1}{2} \ln 2 \right)$$

$= 5/4 + 1/2 \ln 2 - 1/2 \ln 3$

$= 5/4 + 1/2 \ln 2/3$ } 0,5

Exo 8 : (3)

1) $(\cos t)^4 = \left(\frac{e^{it}+e^{-it}}{2}\right)^4$

$= \frac{1}{16}\left(e^{it}+e^{-it}\right)^4$

$= \frac{1}{16}\left[(e^{it})^4 + 4(e^{it})^3 e^{-it} + 6(e^{it})^2(e^{-it})^2 + 4e^{it}(e^{-it})^3 + e^{-4it}\right]$

$= \frac{1}{16}\left[e^{4it} + 4e^{it} + 6 + 4e^{-it} + e^{-4it}\right]$

$= \frac{1}{16}\left[2\cos(4t) + 4\times 2\cos(t) + 6\right]$

$$\boxed{(\cos t)^4 = \frac{\cos(4t)}{8} + \frac{1}{2}\cos(t) + 3/8}$$ } 1

2) $x(t) = a\cos(t) \Rightarrow dx = -a\sin t\, dt$

$y(t) = b\sin(t) \Rightarrow dy = b\cos t\, dt$

$$\int_{(C)} x^3\, dy = \int_{t=0}^{2\pi} (a\cos t)^3 (b\cos t)\, dt$$ } 0,75

$= a^3 b \int_{t=0}^{2\pi} \cos^4 t\, dt = a^3 b \int_{t=0}^{2\pi} \left(\frac{\cos(4t)}{8} + \frac{1}{2}\cos(t) + \frac{3}{8}\right) dt$

$= a^3 b \left[\frac{1}{8}\frac{\sin(4t)}{4} + \frac{1}{2}\sin(t) + \frac{3}{8}t\right]_{t=0}^{t=2\pi}$

$= a^3 b \left[\frac{1}{8}\cdot\frac{\sin(8\pi)}{4} + \frac{1}{2}\sin(2\pi) + \frac{3\cdot 2\pi}{8} - 0\right]$

$$\boxed{\int_{(C)} x^3 dy = \frac{a^3 b \cdot 3\pi}{4}}$$ } 0,75

3) On pose $P = 0$ et Q / $\frac{\partial Q}{\partial x} = x^2 \Rightarrow$ par exemple $Q(x,y) = \frac{x^3}{3}$

Ainsi d'après la formule de Green Riemann :

$$\iint_{\mathcal{E}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_{(C)} P\, dx + Q\, dy = \frac{1}{3}\int_{(C)} x^3\, dy$$

$$= \frac{1}{3}\cdot\frac{a^3 b}{4}\cdot 3\pi = \frac{a^3 b \pi}{4}$$

0,75

Exo 9 : (2)

1) $\omega = \frac{1}{y}\,dx + \left(1 - \frac{x}{y^2}\right) dy$

$P(x,y) = 1/y \qquad Q(x,y) = 1 - x/y^2$

$\frac{\partial P}{\partial y} = -1/y^2 \qquad \frac{\partial Q}{\partial x}(x,y) = -1/y^2$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ 0,5

De ⊕ U est simplement connexe $\Rightarrow$ ω est donc exacte

Il existe donc u / $\omega = du$ (u étant C^1)

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = 1/y \\ \frac{\partial u}{\partial y} = 1 - x/y^2 \end{cases}$$ 0,25

$$\Rightarrow \begin{cases} u(x,y) = x/y + B(y) \text{ avec } B \text{ fonct° } C^1 \\ -x/y^2 + B'(y) = 1 - x/y^2 \end{cases}$$

$$\Rightarrow \begin{cases} u(x,y) = x/y + B(y) \\ B'(y) = 1 \quad \Rightarrow B(y) = y + cst \end{cases}$$

Les primitives de ω sont donc de la forme

$u(x,y) = x/y + y + cst \qquad cst \in \mathbb{R}$ 0,5

2) ω étant exacte :

$$\int_{(C)} \omega = u(B) - u(A)$$
$$= u((9,5)) - u((2,1))$$
$$= 5 - (2+1) = 2$$

0,75

Ex 5 (1,75) On a, pour $D = \{ 0 \le z \le 4(x^2+y^2) \le 1 \}$,

$V = \iiint_D 1 \, dx\, dy\, dz$ (0,25) et en coordonnées cylindriques,

$x^2+y^2 = r^2$, donc $(x,y,z) \in D$ donne $0 \le z \le 4r^2 \le 1$

donc $0 \le r \le \frac{1}{2}$ et $0 \le z \le 4r^2$ (et $\theta \in [0, 2\pi]$) (0,5)

$\mathcal{D}$

$V = \iiint_{\mathcal{D}} r \, dr\, d\theta\, dz$ (0,25)

$= \int_0^{1/2} r \int_0^{4r^2} \int_0^{2\pi} d\theta\, dz\, dr$ (0,25)

$= 2\pi \int_0^{1/2} r \int_0^{4r^2} dz\, dr = 2\pi \int_0^{1/2} 4r^3 dr = 2\pi \left[r^4 \right]_0^{1/2} = \frac{\pi}{8}$ (0,5)